1 Illustrations from the standard toolbox

> *"Thoughts can be steered in different ways."*
>
> – Stanislaw Ulam

Here we illustrate some classical methods that can be used to solve a variety of questions in combinatorics. The contents of this chapter would serve as a precursor to many later chapters in the book.

1.1 Permutations and cycle decompositions

By a permutation of a set S we mean a bijective function from S onto S itself. An elementary counting argument would say that if S is a finite set with $|S| = n$ then the number of permutations of S is $n!$. In particular, the number of permutations of a set is much larger than the size of the set. An element x of a set S is said to be a ***fixed point*** for a permutation σ of S if $\sigma(x) = x$.
An interesting and often useful way to represent permutations is in terms of their cycle decompositions. For a set $S = \{b_1, ..., b_n\}$ and for any $A = \{a_1, ..., a_m\} \subseteq S$ we write $(a_1, a_2, ..., a_m)$ to denote the permutation σ which satisfies $\sigma(a_j) = a_{j+1}$ for all $j \in \{1, ..., m\}$ with the convention that $a_{m+1} = a_1$, and $\sigma(x) = x$ for all $x \notin A$. Such a permutation σ is known as an ***m-cycle***.

✍ **Exercise 1.1.1.** Find out the number of m cycles if $|S| = n$.

When several permutations are written consecutively, they are performed one at a time from *right to left* (as consistent with ordinary function composition). For example, if $S = \{1, 2, 3, 4, 5, 6, 7\}$ then $(1, 3)(1, 2)$ is simply the permutation which maps 1 to 2, 2 to 3 and 3 to 1 and fixes all other elements of S. One can thus write $(1, 3)(1, 2) = (1, 2, 3)$. $2-$cycles are also known as ***transpositions***.

1.1.1 Parity of permutations

While speaking about permutations of a finite set, it is often useful to view a finite set S of size $|S| = n$ as the set $\{1, ..., n\} \subset \mathbf{N}$. Thus, we would speak about permutations of such sets only.
Given a permutation σ consider the sign of the product

$$Q(\sigma) = \prod_{1 \le i < j \le n} (\sigma(j) - \sigma(i)).$$

Note that $Q(\sigma) \neq 0$. If $Q(\sigma) > 0$ we say that σ is an ***even permutation***, and if $Q(\sigma) < 0$ then we say that σ is an ***odd permutation***.
If id denotes the ***identity permutation*** which fixes every element of S, then clearly $Q(\mathsf{id}) > 0$. We have the following result which allows us to compute parity of permutations in a much easier way.

Proposition 1.1.2

For any permutation σ of a finite set S and any two distinct elements a, b of S one has:
$$Q(\sigma \circ (a, b)) = -Q(\sigma).$$

Proof. Without loss of generality we may assume that $a < b$. Then notice that for any c satisfying $a < c < b$ the term containing a, c and the one containing b, c in the product $Q(\sigma \circ (a, b))$ has opposite signs as the terms containing a, c and b, c in the product $Q(\sigma)$. Furthermore, the terms which contain a, b in the product $Q(\sigma \circ (a, b))$ and $Q(\sigma)$ also have an opposite signs. Finally, notice that the absolute value of each term in the product $Q(\sigma \circ (a, b))$ remains the same as the absolute value of the corresponding term in the product $Q(\sigma)$. Thus the claim follows. □

This and the ensuing discussion above shows that each transposition is an odd permutation, and permutations which are product of an even number of transpositions are even permutations and those which are products of an odd number of transpositions are odd permutations.

✍ **Exercise 1.1.3.** Prove that every permutation can be expressed as a product of transpositions.

The following solution to a famous puzzle from the late nineteenth century, often attributed to Sam Loyd, shows how considering parity of permutations can be useful in solving problems which are apparently tricky.

Example 1.1.4

Sam Loyd's 15 puzzle is played with 15 cells labelled as $1, 2, ..., 14, 15$ and arranged in a 4×4 grid in some fashion with one empty cell. The aim is to get to the following configuration by sliding tiles adjacent to the empty location to fill it up and create an empty location at the position of the slid tile and continuing to do so until the following configuration is reached.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	empty

Show that one cannot reach the above configuration from the following one by using only legal moves as described above.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	empty

Solution. The problem can be solved by a simple application of parity of permutations. We write permutations in their cycle notation and discuss in terms of them. A transposition is a 2 cycle (i.e., it switches between some value a, b and keeps everything else fixed). Assume to the contrary that reaching the desired configuration from the given

one is possible by making only a finite number of legal moves.
Note that each move is basically a transposition and hence an odd cycle, however since the empty cell ends up in the same position in both the configurations so we must use an even number of moves.
So, our assumption leads to the conclusion that we can apply an even number of transpositions to get to the desired configuration.
However, notice that any such permutation which is a product of an even number of transpositions must be an even permutation, but the two given configurations vary by only a transposition which is an odd permutation, hence the claim. □

One may like to know about all the arrangements of the 15 squares in the Sam Loyd puzzle for which one can reach the desired configuration using only legal moves. However, this is not a very easy problem to solve. For details one may read [14].

1.2 Pigeonhole principle

We begin with a discussion on the infamous pigeonhole principle[1] which among other things demonstrates how simple ideas can be used to achieve big results in mathematics.

Lemma 1.2.1 (Dirichlet's box principle)
Let n be a positive integer and let $A_1, A_2, ..., A_n$ be n disjoint sets of integers. Then if the union of these sets, namely $\cup_{i=1}^{n} A_i$ has more than n elements then there is at least one $i \in \{1, 2, ..., n\}$ such that A_i contains more than 1 element in it.

The cautious reader will immediately notice that the condition of the A_j s being disjoint is unnecessary. But we mention it in the statement of the theorem because the popular folklore version[2] of it includes the condition of being disjoint.
Here is a generalisation of this principle which can be proved easily.

Theorem 1.2.2 (A generalised pigeonhole principle)
Suppose n is a positive integer and let $k_1, k_2, ..., k_n$ be positive integers. We define $k = 1 + k_1 + k_2 + ... + k_n - n$. If we divide $m \geq k$ many objects in n boxes which are labelled as $\beta_1, \beta_2, ..., \beta_n$ then there exists at least one $i \in \{1, 2, ..., n\}$ such that the box labelled β_i contains at least k_i many objects.

Proof. On the contrary, if we assume that for each $i \in \{1, 2, ..., n\}$ the box labelled β_i

[1] which is sometimes also known as Dirichlet's box principle
[2] the popular version of the theorem (which also justifies the name pigeonhole principle) is stated as *if there are n pigeonholes and more than n letters to be placed in these pigeonholes, then at least one pigeonhole recieves more than one letter.* And as evident, one letter cannot be placed in two or more distinct holes.

has $\leq k_i - 1$ number of objects inside it, then

$$\begin{aligned} k \leq m &= \text{total number of objects} \\ &= \sum_{i=1}^{n} \text{the number of objects in box labelled } \beta_i \\ &\leq \sum_{i=1}^{n} (k_i - 1) = k - 1 \end{aligned}$$

which yields a contradiction, thus refuting our assumption and completing the proof. □

There are various interesting consequences of Dirichlet's box principle. Let us present some solved questions as examples.

Example 1.2.3

Suppose n is an odd positive integer not divisible by 5. Prove that there is at least one multiple of n, with each digit of it (in decimal base) $= 1$.

Solution.

Consider the $n+1$ numbers $a_1 = 1$, $a_2 = 11, ..., a_{n+1} = \overbrace{11 \cdots 1}^{n+1 \text{ times}}$, and the remainders they leave when divided by n. Now, upon division by n, there are only n possible remainders, namely $0, 1, ..., n-1$.
So, by Dirichlet's box principle, there exists $i, j \in \{1, 2, ..., n+1\}$ with $i < j$ such that a_i, a_j leave the same remainder upon division by n. This means, $n | a_j - a_i$.
Now, consider the number $a_j - a_i$ written in decimal base.
The leftmost $j - i$ digits in it are all $= 1$ and the remaining digits are all $= 0$.
Thus $a_j - a_i = a_{j-i} \cdot 10^i$. This means, $n | a_{j-i} \cdot 10^i$.
But since it is given that n is not divisible by 5 and n is odd, so the greatest common divisor of $10, n$ is $= 1$. Hence the greatest common divisor of n and 10^i is $= 1$.
This implies, $n | a_{j-i}$, thus completing the solution. □

This was a routine application of Dirichlet's box principle. Before moving to the next section, let us see another solved application of Dirichlet's box principle. This example (below) will also be discussed and proved in a slightly different way, using a tool which we shall discuss in a later chapter of the document.

Example 1.2.4 (Poland, Second Round, 2003, Day 2)

If p is a prime number > 3, then prove that there are integers x, y, k satisfying $0 < 2k < p$, and the equation $kp + 3 = x^2 + y^2$.

Solution.
Since p is an odd prime number, so a square number z^2 can take one out of $\frac{p+1}{2}$ many values modulo p. So, by Dirichlet's box principle there will be integers x, y satisfying $x^2 + y^2 \equiv 3 \pmod{p}$. And, since we are working with remainders so, we may safely assume that $0 \leq x \leq y \leq p - 1$.
Furthermore, if $y > \frac{p}{2}$, then we may replace y by $p - y$. This allows us to assume without losing any generality that,

$$0 \leq x \leq y \leq \left\lfloor \frac{p}{2} \right\rfloor < \frac{p}{2}$$

(where the last inequality comes from the fact that p is an odd prime number > 3)
Then, we get for some $k \in \mathbb{Z}$ we have $x^2 + y^2 = kp + 3$.
Now, to prove the inequality with k, notice that since 3 cannot be written as a sum of two perfect squares, so $k \neq 0$. And this means $2k > 0$.
Now, notice that the bounds on x, y imply $x^2 + y^2 < \frac{p^2}{2}$.
This means, $kp < \frac{p^2}{2} \implies 2k < p$. We thus get, $0 < 2k < p$.
This completes the solution. $\square$

There are a lot of other beautiful applications of this principle. Below are some fun exercises left for the reader to try.

✍ **Exercise 1.2.5.**

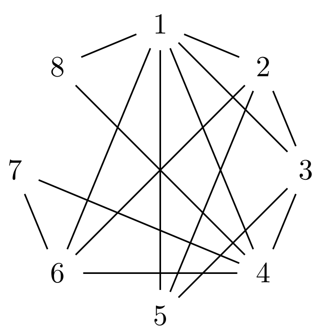
1. Let n be a positive integer, and S be a subset of $\{1, \ldots.., 2n\}$ of size $|S| \geq n + 1$. Show that there must exist $a, b \in S$, such that $a|b$.

2. Let $x \in \mathbf{R}$, and $n \in \mathbf{N}$. Show that there exists integers p, q with $1 \leq q \leq n$, satisfying $\gcd(p, q) = 1$, and $\left|x - \frac{p}{q}\right| \leq \frac{1}{nq}$. Can the inequality be made strict?

3. Filothei has drawn seven points inside or on the boundary of a triangle of unit area. Her sister Anastasia thinks that she can always find three points among these given points such that they are collinear or form a triangle of area $\leq \frac{1}{4}$. Is Anastasia correct?

1.3 A look at graphs

Loosely speaking a ***graph*** is a collection of points known as ***vertices*** (or ***nodes***) some of which are joined using lines called ***edges***. Thus a graph G can be expressed as pair $G = (V, E)$ of a set V of vertices of G, and a set E of edges of V such that $E \subseteq$ the set of two element subsets of V; here we exclude the possibility of a graph to have ***loops*** which are simply edges joining a vertex to itself, and ***multi-edges*** which are basically multiple edges between the same two vertices. The graphs we consider are sometimes also known as ***simple graphs*** to emphasize upon the fact that we exclude the possibility of multi-edges and loops in the graph.
Also unless specified otherwise we shall consider graphs with a finite vertex set (and hence a finite edge set). Following is an illustration of a graph having 8 vertices.

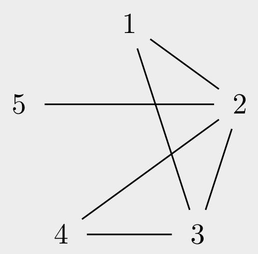


Sometimes it may be convenient to give direction to the edges so that now they are represented by ordered pairs (v, u) of vertices of G, and in such a case we call the graph G ***directed***, and the edges of a directed graph are often known as ***arcs***.

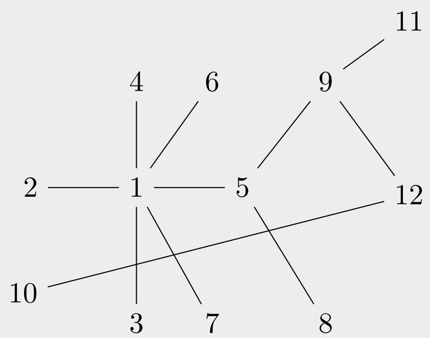
Unless mentioned otherwise our graphs will be ***undirected***, that is to say the graphs will be not directed.
Two vertices v, u of a graph G are said to be ***adjacent*** if there is an edge joining v and u in the graph G. Adjacent vertices are also called ***neighbours***.
For any vertex v of a graph G, we define the ***degree*** of v to be the number of neighbours of v, and is denoted by $\deg(u)$ (some authors use the notation $d(v)$ to denote the degree of a vertex v in a graph); when it is necessary to specify the graph in which the degree is counted then we write $\deg_G(v)$ to denote the degree of vertex v in the graph G (as we shall see such situations arise when we consider working on a smaller part of G or a "subgraph" of G in which the degrees of certain vertices of G change). A vertex whose degree is 0 is called an ***isolated vertex***.

✍ **Exercise 1.3.1.** For each of the following graphs G_1, G_2 :

$G_1 =$



$G_2 =$



Find out the degree of each vertex of this graph, and the neighbourhood of each vertex of the graph.

We have the following useful result:

Lemma 1.3.2
If G is a graph with vertex set V, and edge set E then

$$\sum_{u \in V} \deg(u) = 2|E|.$$

Proof. We begin with the elementary observation that each edge in a simple finite graph has exactly two distinct endpoints. Thus it follows that in the sum $\sum_{u \in V} \deg(u)$, each edge of G gets counted exactly twice (the edge joining the vertices v, w gets counted in $\deg(v)$ and $\deg(w)$), so that one has

$$\sum_{u \in V} \deg(u) = 2|E|.$$

□

Following is an exercise the reader is suggested to try for understanding the above proof better.

✍ **Exercise 1.3.3.** If we have a graph G with a finite vertex set V, without any loops but possibly having multi-edges, then is it necessarily true that:

$$\sum_{u \in V} \deg(u) = 2|E|.$$

The following exercise demonstrates a typical use of Lemma 1.3.2.

✍ **Exercise 1.3.4.** Suppose there are n people who gathered on some occasion. If any two of them shake hands exactly once, then how many handshakes take place?

A ***walk*** from a vertex v_1 to a vertex v_n in a graph G is a sequence of edges and vertices $(v_1, e_{1-2}, v_2, ..., v_{n-1}, e_{(n-1)-n}, v_n)$ of G such that the endpoints of e_j are v_j, v_{j+1} for each $j \in \{1, ..., n-1\}$. The walk is called ***closed*** if $v_1 = v_n$, and ***open*** otherwise.
A ***trail*** is a walk in which all edges are distinct.
A ***path*** is a walk in which all vertices (and hence all edges) are distinct.

We have similar notions for directed graphs as well which take into account the direction of the arcs.

A graph G is said to be ***connected*** if for any two vertices v, u in G there is a path from v to u.

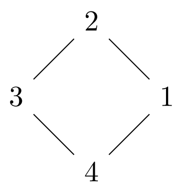
✍ **Exercise 1.3.5.**

i. If G is a connected simple graph having n vertices, then show that it must have at least $n-1$ edges.

ii. Suppose G is a graph on n vertices such that for any two non-adjacent vertices u, v of G one has $\deg(u) + \deg(v) \geq n$.[a] Show that G must be connected.

[a] Such graphs are known as ***Ore's graph***.

A graph G is said to be complete if any two distinct vertices of G are joined by an edge. We write K_n to denote the complete graph on n vertices. It is easy to see that the number of edges in K_n is $\binom{n}{2}$.
Another type of graphs which often makes it presence are the cycles. For a positive integer n, the $n-$ ***cycle*** is the graph obtained by drawing a $n-$gon with the usual vertices and edges. Thus for instance a $4-$ cycle would be as shown below:



Another important notion is that of subgraphs. Let G be a graph with vertex set V_G and edge set E_G; by a ***subgraph*** of G we mean a graph H such that the vertex set V_H of H is a subset of the vertex set V_G of G, and the edge set V_H of H is a subset of the edge set E_G of G; sometimes it is specifically also mentioned that the edges of H have their endpoints in the vertex set V_H of H, although this is automatically guaranteed by the condition that H is a graph in its own right. If H is a subgraph of G, then G is sometimes referred to as a ***supergraph*** of H.

✍ **Exercise 1.3.6.** Show that a graph can be partitioned into connected subgraphs with pairwise disjoint vertex sets.
These connected subgraphs are called ***connected components*** of the original graph.

A very interesting class of graph properties are formed by those properties which are also true for all its subgraphs (resp. supergraphs), and these are called ***subgraph invariant properties*** (resp. ***supergraph invariant properties***). Subgraph invariant properties and supergraph invariant properties are related in the obvious way: if having a certain feature is subgraph invariant, then not having that feature is supergraph invariant. As one would expect, such properties can often be used to conclude results about bigger graphs by working on smaller graphs. Below we discuss an example.

Example 1.3.7

[Mantel's theorem] If G is a simple graph on n vertices and having no $3-$cycles (as subgraphs), then prove that the number of edges in G is $\leq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Solution. We induce on n. The verification of the base case for $n \in \{1, 2, 3\}$ is left to the readers as an elementary exercise. So suppose $N \geq 3$ be a natural number such that for each positive integer $n \leq N$, the result holds true Now, let G be a simple graph on $N+1$ many vertices. Either G has no edge, in which case the result is obvious, otherwise we can find two adjacent vertices v, u of G, and let G' be the subgraph of G obtained by removal of the vertices u, v and the edges incident to them from G. Then, notice that G' has $N+1-2 = N-1$ vertices. Also notice that the property of not having a $3-$cycle is subgraph invariant because if a graph $\mathcal{G}$ does not have any $3-$cycle then none of its subgraphs can contain a $3-$cycle, whence by induction hypothesis it follows that the number of edges in G' is $\leq \left\lfloor \frac{(N-1)^2}{4} \right\rfloor$. By Dirichlet's box principle it also follows that the number of edges incident to at least one of u, v is $\leq (N+1) - 1 = N$. Thus the number of edges in G is $\leq$ the number of edges in G' + N which in turn is

$$\leq \left\lfloor \frac{(N-1)^2}{4} \right\rfloor + N \leq \left\lfloor \frac{(N+1)^2}{4} \right\rfloor.$$

This finishes the inductive step, thus completing the solution.

□

The example discussed above is of an "extremal" flavour; we aimed to extremise something and we will later see how we can show that the bound in Example 1.3.7 is sharp. But before that let us talk about another related concept of ***extremal graphs***. In combinatorics it is often desired to study extremal properties of graphs (especially the maximum number of edges) with a fixed number of vertices which does not have a particular graph as a subgraph. Below we discuss an example of this flavour.

Example 1.3.8

If G is a simple graph on n vertices not containing a $4-$cycle, then prove that the number of edges in G is $\leq \frac{n}{4} \cdot (1 + \sqrt{4n-3})$.

Solution. Let $G = (V, E)$ be a simple graph on n vertices without containing any $4-$cycle as a subgraph. Define

$F_G := \{(u, v, w) \in V^3 :$ there is an edge joining $u, v,$ and an edge joining v, w and $u \neq w\}$;

elements of F_G are often called ***forks*** in G.
An elementary computation gives us $|F_G| = \sum_{u \in V} \deg(u) \cdot (\deg(u) - 1)$.
Now we notice that since G does not contain any $4-$cycle as its subgraph, so if $(u, v, w), (u, v', w) \in F_G$, then $v = v'$, because otherwise (u, v, v', w) would give us a $4-$cycle in G. This simple observations leads us to the conclusion that $1|F_G| \leq n(n-1)$. Putting all this together we see that

$$n(n-1) \geq \sum_{u \in V} \deg(u) \cdot (\deg(u) - 1) \geq \frac{1}{n}\left(\sum_{u \in V} \deg(u)\right)^2 - \sum_{u \in V} \deg(u) = \frac{(2|E|)^2}{n} - 2|E|.$$

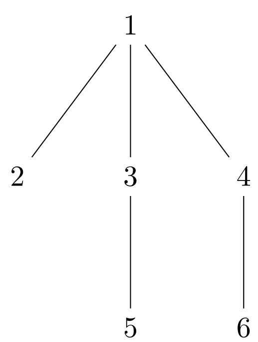
From this one obtains: $|E| \leq \frac{n}{4} \cdot (1 + \sqrt{4n-3})$, as required.

□

Notice that we never utilised the subgraph invariant feature of the property of not having a $4-$cycle anywhere in the above solution. The reader is suggested to try solving the following exercise.

✍ **Exercise 1.3.9.** Give an alternate solution to Example 1.3.8 by exploiting the subgraph invariance feature of the property of not having a $4-$cycle.

Graphs which do not contain cycle as subgraphs are called ***acyclic***. Another important example of graphs are trees. A ***tree*** is a connected acyclic graph (i.e., a connected graph without any cycles).



✍ **Exercise 1.3.10.** Let G be a simple acyclic graph on a finite number of vertices. Show that it must have at least one vertex whose degree is ≤ 1. Can it have exactly one vertex whose degree is $= 1$?

The figure above shows a tree on 6 vertices. Find out the number of edges in this graph. Draw any other tree on 6 vertices and count its number of edges. (Aren't they the same?) In general we have an interesting invariance for the number of edges in a tree with a fixed finite number of vertices. But before that an exercise.

✍ **Exercise 1.3.11.** Show that the property of being a tree is not subgraph invariant, but removing a vertex whose degree is one from a tree (which exists by Exercise 1.3.10 and the fact that a tree is connected) gives a subgraph which is also a tree in its own right.

Now we come to an interesting result regarding the invariance of the number of edges in a tree with a fixed finite number of vertices.

Lemma 1.3.12

Let T be a tree on n vertices. Then the number of edges in T is exactly $= n - 1$.

Proof. We induce on n. Notice that the base case $n = 1$ is vacuous while the case $n = 2$ is obvious. Now suppose N is a positive integer such that for all positive integers $k \leq N$, it is true that whenever T_k is a tree having k vertices then the number of edges in T_k is $= k - 1$. Let T_{N+1} be a tree having $N + 1$ vertices.
From Exercise 1.3.10 it follows that T_{N+1} has at least one vertex of degree ≤ 1, and since T_{N+1} is a tree so in particular it is connected and hence every node of T_{N+1} has degree ≥ 1, and thus it follows that there exists at least one vertex u in T_{N+1} whose degree $\deg(u) = 1$. Let T' be the subgraph of T_{N+1} obtained by deleting this vertex u and any edge incident to it from T_{N+1}. Then by Exercise 1.3.11 it follows that T' is a tree on $(N + 1) - 1 = N$ many vertices, whence by induction hypothesis it follows that the number of edges in T' is $N - 1$. Finally we notice that the number of edges that got removed due to the deletion of the vertex u from T_{N+1} is one, and so T_{N+1} has $(N - 1) + 1 = N = (N + 1) - 1$ many edges as required. Thus the claim follows. □

The following exercise is in spirit of Lemma 1.3.12, and in some sense sheds light on an equivalent definition for trees (see Exercise 1.3.14).

✍ **Exercise 1.3.13.** Prove that any graph with n vertices and k many edges has at least $n - k$ many connected components.
(Hint: decompose the given graph into its connected components and observe that an edge is a part of exactly one of the connected components of the graph; also Lemma 1.3.12 implies that if the i−th connected component has n_i many vertices then it must have $\geq n_i - 1$ many edges (why?); conclude the claim.)

One can define directed trees by giving directions to the edges; they arise frequently in designing algorithms based on graphs; we will discuss them in a later chapter.

✍ **Exercise 1.3.14.**

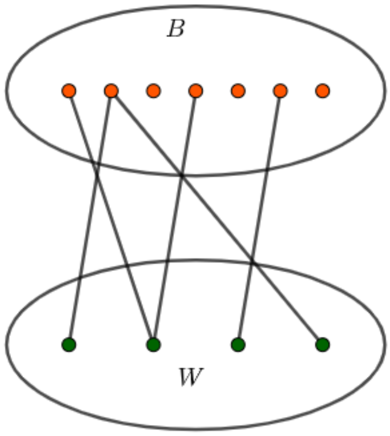
i. Is it necessarily true that any simple connected graph on n vertices and having $n - 1$ edges is acyclic and hence a tree?

ii. Is it necessarily true that any simple acyclic graph on n vertices and having $n - 1$ edges is connected and hence a tree?

Another notion often studied in graph theory is that of the complement of a graph. Given a graph $G = (V_G, E_G)$ on a vertex set V_G and edge set E_G the ***complement*** of G is the graph G^c having the same vertex set V_G as that of G, but whose edge set is the complement of the edge set E_G of G, that is to say that, two distinct vertices $u, v \in V_G$ are adjacent in G^c if and only if they are not adjacent in G. It is easy to see that $(G^c)^c = G$.

✍ **Exercise 1.3.15.** If G is a finite simple graph which is not connected, then is it necessarily true that its complement G^c is connected?

Another important class of graphs are bipartite graphs or more generally k partite graphs. A graph G is said to be a ***k partite*** graph if we can color the vertices of G with k many distinct colors in such a way that no two adjacent vertices receive the same color. $2-$partite graphs are also called ***bipartite*** graphs.
In other words, a graph G is bipartite if we can distinguish its vertices into two sets $B(G), W(G)$ in such a way that each edge of G has one of its endpoints in $B(G)$ and the other end point in $W(G)$. Following is a pictorial illustration of a bipartite graph.



Bipartite graphs enjoy several properties.

Proposition 1.3.16

A finite simple graph G is bipartite if and only if it does not contain any cycle of odd length.

Proof. Suppose G is a finite simple bipartite graph on vertex set V and with E as its edge set. Let V admit the partition $V = B \cup W$ with $B \cap W = \emptyset$ and every edge in E having one end point in B and the other in W.
We would show that G has no cycle of odd length.
Suppose C is a cycle in G having n vertices, and let the vertices of C be labelled as $v_1, ..., v_n, v_{n+1} := v_1$ where v_i and v_{i+1} are adjacent for all $i \in \{1, ..., n\}$. Without loss of generality let $v_1 \in B$. Then $v_2 \in W, v_3 \in B, ...$ in general $v_{2k} \in W$ for all $k \in \mathbf{N}$ such that $2k \in \{1, ..., n\}$ and $v_{2\ell+1} \in B$ for all integers ℓ for which $2\ell + 1 \in \{1, ..., n + 1\}$, since $v_{n+1} \in B$, so $n + 1$ must be odd and hence n must be even; since C was any arbitrary cycle in G, this shows that G does not contain any cycle of odd length.

The converse is left as an exercise.

Exercise 1.3.17. Prove the remaining direction of Proposition 1.3.16.

This concludes the proof. □

Another important class of graphs are the ones' whose vertices all have the same degree, they are called ***regular graphs***. More precisely, a graph G is said to be ***m regular*** if all the vertices of G have degree m.

Proposition 1.3.18

Let G be a simple m regular bipartite graph on a finite vertex set $V(G)$, for $m > 0$. Then $|V(G)|$ must be even.

Our proof of Proposition 1.3.18 will be done using a very simple yet useful proof/derivation technique in combinatorics called *double counting*; the idea is based on the fact that if we count the elements of a set A in two different ways and arrive at two different expressions for $|A|$ then these two expressions must be equal in terms of values they take.

Proof. Let $E(G)$ denote the edge set of G, and let the vertex set $V(G)$ of G be partitioned as $V(G) = B(G) \cup W(G)$ where $B(G) \cap W(G) =$ and each edge of G has one end point in $B(G)$ and the other in $W(G)$. Then counting the number of edges by counting their end points which lie in $B(G)$ and $W(G)$ respectively one has

$$|E(G)| = \sum_{b \in B(G)} \deg(b) = |E(G)| = \sum_{w \in W(G)} \deg(w),$$

this along with the fact that G is m regular for $m \neq 0$ so that $\deg(v) = m \neq 0$ for each $v \in V(G)$, one obtains $\sum_{b \in B(G)} \deg(b) = m|B(G)|$, and $\sum_{w \in W(G)} \deg(w) = m|W(G)|$, thus $m|W(G)| = m|B(G)|$ and so cancellation of the non-zero factor m from both sides gives us $|W(G)| = |B(G)|$, so that $|V(G)| = |B(G)| + |W(G)| = 2|B(G)|$ is twice an integer and hence an even number. □

The following fun problem from a past Arithmathon contest demonstrates a manifestation of the simple fact discussed in Proposition 1.3.18.

Example 1.3.19 (Arithmathon, category: J2, topic: combinatorial mathematics)

A new chess-piece called *super-knight* is introduced. It moves as follows:

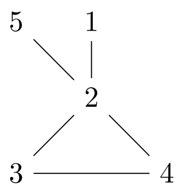
* whenever summoned the user playing this piece needs to choose two positive integers a, b of opposite parities;
* the super-knight is then allowed to move a steps towards north or south along the column where it was situated during the time it was summoned and then it moves b many steps towards east or west along the row reached after travelling a total of a vertical steps;
* it occupies the cell reached after travelling the total $a + b$ steps according to the above rules, thus killing any unit occupying this cell at the time a super-knight reaches there.

Suppose $N \in \mathbf{N}$ is a positive integer such that for some positive integer $m > 0$, we can place N many super-knights on a bi-infinite chessboard in such a way that each super-knight is at an attacking position for exactly m other super-knights (among the ones' placed by us). Show that N must be even.

Solution. We begin with the following observation: each time a super-knight it summoned it moves a total of an odd number of steps, and thus the color of the cell occupied by it before and after the move are distinct. This in particular implies that, no two super-knights on cells of the same color can be at an attacking position for each other.
Now returning back to the problem, suppose the super-knights are labelled as $\mathfrak{K}_1, \ldots\ldots, \mathfrak{K}_{\mathfrak{N}}$.

We place these super-knights on a bi-infinite chessboard in such a way that each of them attack exactly m other. Let $\mathcal{B}, \mathcal{W}$ respectively denote the set of super-knights placed on black and white cells. Then, as noted above no two super-knights in $\mathcal{B}$ attack each other, and no two super-knights in $\mathcal{W}$ attack each other.
Now let G be the graph on vertex set $V(G) := \{v_1, \ldots\ldots, v_N\}$ where two vertices v_i, v_j are adjacent if and only if the super-knights $\mathfrak{K}_i$ and $\mathfrak{K}_j$ are in attacking positions for each other. Then, since no two super-knights in $\mathcal{B}$ attack each other and no two super-knights in $\mathcal{W}$ attack each other, so it follows that for the following subsets $B(G) := \{v_i \in V(G) : \mathfrak{K}_i \text{ is placed on a black cell }\}$ and $W(G) := \{v_i \in V(G) : \mathfrak{K}_i \text{ is placed on a white cell }\}$ each edge of G has exactly one end point on $B(G)$ and the other end point in $W(G)$. Thus, G is a bipartite graph. Finally we notice that since each super-knight attacks exactly m other super-knights in this configuration so each vertex of G has degree $= m$. Thus, G is a m regular bipartite graph. This implies that $N = |V(G)|$ is even (thanks to Proposition 1.3.18).
This completes the solution. $\square$

Two other related and important notions that often arise in graph theory are those of Eulerian and Hamiltonian paths. A graph $\mathcal{H}$ is ***Hamiltonian*** if there is a path which contains all the vertices of $\mathcal{H}$ exactly once. Thus for instance the following graph



is not Hamiltonian. It is in general a NP hard problem to determine whether a given graph is Hamiltonian or not. However the following result due to Oystein Ore helps us to conclude that a graph is Hamiltonian if each pair of non-adjacent vertices have a sufficiently large sum of degrees. More precisely one has the following:

Theorem 1.3.20 (Ore's theorem)
If G is a simple graph on n vertices such that for each pair (u, v) of non-adjacent vertices of G one has $\deg(u) + \deg(v) \geq n$, then G is Hamiltonian.

Proof. We know that such a graph G (which is also known as an Ore's graph) must be connected, thanks to Exercise 1.3.5 (part ii). Now, assume to the contrary that the claim is false. Let G be an Ore's graph which is not Hamiltonian and which has the maximum number of edges among all non-Hamiltonian Ore's graphs. Then G must contain a path ρ which visits each vertex of G exactly once, for otherwise we could simply add new edge to a maximum length path visiting each vertex once to make it visit new vertices whilst keeping it a non-Hamiltonian Ore's graph. Thus, we can label the vertices as $v_1, \ldots., v_n$ so that the path ρ if started at v_1 visits the vertices in the respective order as $v_1, v_2, \ldots., v_n$. Then, notice that v_n cannot be adjacent to v_1 because otherwise that would give us a Hamiltonian cycle in G thus refuting our assumption. So, one has $\deg(v_1) + \deg(v_n) \geq n$; also the vertices adjacent to at least one of v_1, v_n must belong to the set $\{v_2, \ldots., v_{n-1}\}$, thus by pigeonhole principle it follows that there is some $2 \leq i - 1 \leq n$ such that the vertex v_i is adjacent to v_1 and the vertex v_{i-1} is adjacent to v_n; but then the cycle $v_1, \ldots., v_{i-1}, v_n, v_{n-1}, \ldots., v_i, v_1$ is a Hamiltonian cycle in G. Thus, in either case G turns out to be Hamiltonian. This shows us that our assumption is false, whence its negation is true, and this proves Ore's theorem. $\square$

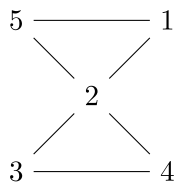
An easy corollary of Ore's theorem is the following interesting result due to Paul Dirac.

Corollary 1.3.21 (Dirac's theorem)

If in a simple graph G on a finite vertex set $V(G)$ each vertex has degree $\geq |V(G)|/2$ then G must be Hamiltonian.

Proof. Take any two non-adjacent vertices u, v of G. Then $\deg(v) \geq |V(G)|/2, \deg(u) \geq |V(G)|/2$, and thus $\deg(u) + \deg(v) \geq |V(G)|$, the claim follows by Ore's theorem. □

A related notion is that of an Eulerian graph. A graph $\mathcal{E}$ is said to be ***Eulerian*** if there is a closed trail which visits each edge of the graph exactly once. Thus for instance the following graph



is Eulerian.

✍ **Exercise 1.3.22.** Is the above graph Hamiltonian?

As opposed to the difficulty in determining whether a graph is Hamiltonian or not, it is easy to determine whether a graph is Eulerian or not.

Theorem 1.3.23

A simple, finite connected graph G is Eulerian if and only if every vertex of G has an even degree.

Proof. For one direction suppose that G is an Eulerian simple finite connected graph. Then there exists a closed trail in G which visits every edge exactly once; let τ be such a trail. Let v be any arbitrary vertex of G. We would show that v has an even degree in G. To this end we enumerate the vertices of G in the order in which they get traversed in the trail τ to get a labelling of the vertices as $g_1, g_2, \ldots\ldots\ldots, g_n, g_1$; notice that we may assign more than one distinct labels to a vertex because the trail τ need not visit all vertices exactly once.

Now we introduce a relation $\rightsquigarrow$ among the vertices of G by saying $u \rightsquigarrow w$ for vertices u, w of G, if and only if there exists $i \in \{1, 2, \ldots\ldots, n\}$ such that $u = g_i$ and $w = g_{i+1}$, where we set the convention $g_{n+1} := g_1$.

Notice that by the conditions on G and the trail τ one has

$$\deg(v) = |\{z \in V(G) \ : \ z \rightsquigarrow v, \text{ OR}, v \rightsquigarrow z\}|.$$

Also notice that, since τ visits each edge exactly once and is closed so, it follows that

$$|\{z \in V(G) \ : \ z \rightsquigarrow v\}| = |\{z \in V(G) \ : \ v \rightsquigarrow z\}|.$$

Finally we notice that since G is simple so there is at most one edge between any two vertices of G and thus one has $\{z \in V(G) \ : z \rightsquigarrow v\} \cap \{w \in V(G) \ : \ v \rightsquigarrow w\} = \phi$, whence

the claim follows.
Now for the other direction suppose G is a simple finite connected graph such that every vertex of G has an even degree. We need to show that G must be Eulerian.
First of all notice that if G has zero edges then there is nothing to prove. So now suppose that G has at least one edge.
Let $W = v_0, \ldots.., v_t$ be the longest walk in G which does not traverse any edge more than once.
First of all notice that W must traverse every edge incident to v_t; otherwise we can just append any un-traversed edge to W to get an even longer walk thus contradicting the maximality of the length of W. Also observe that the condition that degree of each vertex of G is even implies that $v_t = v_0$.
We claim that W must be an Eulerian trail. To this end we notice that in view of the above observations made about W, it suffices to show that there is no edge which is not traversed by W. Assume to the contrary that there is some un-traversed edge. Then since G is connected there must exist a vertex v_i in W which is incident to this un-traversed edge; say the un-traversed edge is $\{u, v_i\}$. Then we can make an even longer walk

$$W' = u, v_i, v_{i+1}, \ldots.., v_t, v_1, \ldots\ldots., v_i, u$$

by appending the edge $\{u, v_i\}$ to the walk W. But this contradicts the maximality of the length of W, thereby implying that our assumption is false. Thus the claim follows.

□

Another way to prove Theorem 1.3.23 is to induce on the number of vertices of the graph G. We leave the details as an exercise for the reader.

✍ **Exercise 1.3.24.** Give an alternative proof of Theorem 1.3.23 by using induction on the number of vertices of G.

Another important notion in graph theory is that of isomorphism of graphs. A graph $G_1 = (V_1, E_1)$ is said to be ***isomorphic*** to a graph $G_2 = (V_2, E_2)$ if there is a function $f : V_1 \to V_2$ such that f is a bijective function, and there is an edge joining a vertex v, to a vertex u in G_1 if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G_2; if such a function f exists then it is called an ***(graph) isomorphism***.

✍ **Exercise 1.3.25.**

i. Show that the property of graphs being isomorphic is an equivalence relation. (That is to say that, if we define a relation $\sim$ on a set of graphs by saying that a graph $G \sim$ a graph H if and only if G is isomorphic to H, then this relation $\sim$ is an equivalence relation.)

ii. Show that if a graph G is isomorphic to a tree (resp. connected graph, cyclic graph, $k-$partite) then G must be a tree (resp. connected graph, cyclic graph, $k-$partite).

iii. If $n \in \mathbf{N}_{\geq 4}$, then is it true that any two trees on n many vertices are isomorphic?

iv. Can you find any (possibly infinite) graph G which is isomorphic to a proper subgraph of itself?

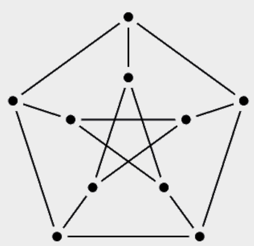
Famously a graph is represented by its ***adjacency matrix***, where the rows and columns of the matrix are labelled according to the vertices of the graph and the $(i, j)-$th entry of the matrix is the number of edges joining the $i-$th vertex to the $j-$th vertex in the graph; there is a subtle issue if the vertex set is not countable in which case there is no standard notion of ordering the vertices, but one can use tools such as the axiom of choice to give the set of all vertices a total ordering and then represent a graph in this fashion.

✍ **Exercise 1.3.26.** Let G_1, G_2 be isomorphic graphs and let P_1, P_2 be the adjacency matrices of G_1 and G_2 respectively. Then is it necessarily true that P_2 can be obtained by permuting the rows of P_1?

Often eigenvalues of the adjacency matrix are studied to derive a lot of important information about the graph itself, thus for instance knowledge of some properties of a given graph can be used to derive certain properties about the set of eigenvalues (which is also known as ***spectrum***) of the graph which leads to some further knowledge about the graph;for several reasons often the greatest eigenvalue of the adjacency matrix is considered while analysing the graph. We won't pursue a discussion on this face of graph theory in this book; the interested reader can refer to any standard textbook on algebraic graph theory.

Two commonly used terminologies in graph theory are cliques and independent sets. For a graph G by a ***clique*** of G we mean a complete subgraph of G, and by an ***independent set (of vertices)*** of G we mean a set U of vertices of G such that no two distinct vertices in U are adjacent in G; often people also use the term *clique* to refer to what we call ***maximum cliques*** which is a clique of G which has the maximum number of vertices among all the cliques of G; another related notion is that of a ***maximal clique*** of G which is a clique $\mathcal{K}$ of G such that there does not exist any vertex v of G which is not a vertex of $\mathcal{K}$ and for which the subgraph H of G obtained by joining v with the vertices of $\mathcal{K}$ which are neighbours of v in G is complete (i.e., we cannot enlarge a maximal clique by adding a new vertex to its vertex set). We will use these terminologies later discussions about graphs.

✍ **Exercise 1.3.27.** In the graph drawn below find the independent sets, cliques, maximum cliques and maximal cliques; this graph is famously known as the ***Petersen graph***.



Graph theory will be discussed in greater details in a later chapter.

Following are some exercises left for the readers to try.

✍ **Exercise 1.3.28.**

1. A graph G is said to be ***self-complementary*** if G is isomorphic to its complement G^c. Show that if n is a positive integer, such that there exists a self-complementary graph on n vertices, then $4|n(n-1)$. Also for each positive integer n for which $4|n(n-1)$, show by a construction or otherwise, that there is a self-complementary graph on n vertices.

2. Give example of a connected simple finite graph G, whose complement G^c is also connected.

3. A ***cut-edge*** in a graph G is an edge of G whose removal from G increases the number of connected components of G.
   i. Let G be a simple finite graph such that every vertex of G has an even degree. Show that G does not have a cut-edge.
   ii. For each positive integer k, show that there is a $2k+1$ regular graph having a cut-edge.
   iii. Give example (if exists) of a $2k+1$ regular graph which does not have a cut-edge.

4. If k is a natural number and G is a k regular bipartite graph with vertex classes X, Y then show that $|X| = |Y|$.
   (Hint: show that it suffices to prove the claim when G is connected; for a finite simple connected bipartite k regular graph G with vertex classes X, Y write down $|X|$ in terms of the number of edges in G and k to conclude the proof.)

5. If G is a finite simple graph, then show that there is a bipartite subgraph B_G of G having at least $|E(G)|/2$ many edges in it.

6. If G is a regular, finite, bipartite graph then show that G does not have any cut-edge.
   (Hint: it is easy to show for $k = 1$; for $k \geq 2$ if G had a cut-edge $\{x, y\}$ then upon removal of this edge from G and taking the connected component of x in this new subgraph we obtain a bipartite graph with one vertex of degree $k-1$ and all other vertices of degree k; obtain a contradiction by summing up the degrees on each vertex class of this bipartite graph and noticing that for a bipartite graph the sum of degrees in each of the vertex classes must be the same and working modulo k.)

1.4 A taste of Ramsey theory

Another important branch of combinatorics with applications to branches beyond mathematics is Ramsey theory. Ramsey theory was initiated by the following famous observation by Frank Ramsey.

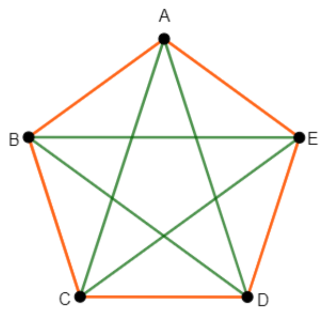
Proposition 1.4.1

If the edges of the complete graph K_6 are colored with just two colors black and white, then there is a monochromatic triangle in this colored graph.

Proof. We label the six vertices of K_6 as A, B, C, D, E, F. Consider vertex A. There are 5 edges in K_6 with A as an endpoint. By pigeonhole principle, at least three of these edges must be of the same color. Without loss of generality let us assume that the edges

joining A with B, C, D are of white color. Then, either the edges CB, BD, DC are all black in color, or there is some edge among them which is white.
If all the edges $(C, B), (B, D), (D, C)$ are black then we obtain a black triangle in this colored K_6 which is formed by the vertices B, C, D.
Otherwise if at least one of these edges are of white color, say for instance if (C, B) is of white color, then A, C, B gives us a white triangle in this colored K_6.
Thus for any coloring of the edges of K_6 using just two colors, there is a monochromatic triangle. □

However the following figure demonstrates a colouring of the edges of K_5 using two colors but without containing any monochromatic triangle.



Thus we conclude that 6 is the smallest positive integer n such that for any arbitrary 2-coloring of the edges of the complete graph K_n on n vertices one can find a monochromatic triangle (i.e., in this case a triangle with all its three edges having the same color).

Now one can ask for the smallest positive integer n such that for any 2 colouring of the edges of K_n one can always find a monochromatic 4 clique, 5 clique, 6 clique, etc..

Definition 1.4.2. For positive integers m, n the ***Ramsey number*** $R(m, n)$ is the smallest positive integer N such that for any colouring of the edges of any simple graph on N vertices using just two colours one can find a monochromatic clique of size n or an independent set of size m.

✍ **Exercise 1.4.3.** Prove that $R(m, n) = R(n, m)$ for all positive integers m, n.

Unlike finding $R(3, 3)$ it is very difficult to find $R(m, n)$ even for slightly larger values of m, n, for instance the value of $R(5, 5)$ is not known till date. The following joke cracked by Paul Erdős echoes on the difficulty of finding higher Ramsey numbers

"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number $R(5, 5)$; we could marshal the world's best minds and fastest computers, and within a year probably calculate the value. However, if the aliens demand the Ramsey number $R(6, 6)$, then instead of trying to find its value we should try to find ways to destroy the aliens before they destroy us."

- Paul Erdős

The following claim provides a recursive bound on the Ramsey number $R(m, n)$ which can be used to derive an upper bound on Ramsey numbers which in particular proves Ramsey's celebrated result that $R(m, n)$ is finite for all positive integers m, n.

Lemma 1.4.4

For all positive integers m, n one has

$$R(m+1, n+1) \leq R(m, n+1) + R(m+1, n). \tag{1.1}$$

Proof. Consider a complete graph G on $R(m+1,n) + R(m,n+1)$ vertices whose edges are arbitrarily coloured using two colours. Let v be a vertex of this graph. We partition the set $V(G) \setminus \{v\}$ into two parts A, B according to the following rule: A contains all and only those vertices of G which are joined to v by a red coloured edge, and B is the set of all and only those vertices of G which are joined to v by a blue coloured edge. Then either $|A| \geq R(m+1,n)$, or $|B| \geq R(m,n+1)$. If $|A| \geq R(m+1,n)$ then either there is a blue clique having $m+1$ vertices in the subgraph of G induced by A, or there is a red clique having $n+1$ vertices in the subgraph of G induced by A. For the former we have a blue clique in G having $m+1$ vertices; for the latter appending v to the red clique having n vertices gives us a red clique having $n+1$ vertices in G. Similarly if $|B| > R(m,n+1)$ one can show that there exists a red clique having $n+1$ vertices or a blue clique having $m+1$ vertices in G. The claim follows. □

The inequality in 1.1 can be used inductively to obtain the following:

Corollary 1.4.5 (Ramsey's theorem)

For all positive integers m, n one has

$$R(m,n) \leq \binom{m+n-2}{m-1} \tag{1.2}$$

In particular $R(m,n)$ is finite for all positive integers m, n.

The conclusion that $R(m,n)$ is finite for all positive integers m, n is known as Ramsey's theorem, named after Frank Ramsey whose work initiated the modern day branch of mathematics called Ramsey theory.

1.5 Bijective strategies

One very useful method of solving problems in combinatorics is to make use of the following result, which essentially says that in order to find the number of elements in a finite set, it is equivalent to count the number of elements after relabelling them in any order. More precisely, we have the following:

Lemma 1.5.1

Let A, B be two finite sets such that there is a bijective function $f : A \to B$. Then the number of elements in A is equal to the number of elements in B.

Proof. Let $|A| = n$, and $A = \{a_1, ..., a_n\}$, then the given conditions imply that the elements of the set B can be enumerated as $B = \{f(a_1), ..., f(a_n)\}$, noting that f is injective proves the claim. □

This elementary result can be used in a variety of instances, and it is often very difficult to foresee the scope for applying this result. Here we present an example of how this result can be used.

Example 1.5.2

Suppose every vertex of a regular 9 gon is colored using red or blue. Prove that, there exist two congruent triangles, both of whose vertices are all red or both of whose vertices are all blue.

Solution. We would call a triangle blue (respectively red) if all its vertices are blue (respectively red). Because there are nine vertices colored using two colors, so at least five vertices must be of the same color. Without loss of generality assume that this color is red, hence there will be at least $\binom{5}{3} = 10$ red triangles. We would show that that there are two congruent red triangles.
Let the vertices of the 9 gon be $v_1, v_2, ..., v_9$ (in clockwise order), and let Γ denote its circumcircle. Since $v_1, ..., v_9$ form a regular 9 gon, so they cut Γ into 9 equal arcs.
We shall call each such arc a *piece*.

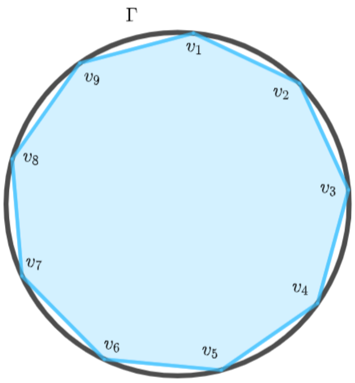


Figure: the regular 9 gon and its circumcircle.

Let $v_iv_jv_k$ be a triangle with $|v_iv_j| \leq |v_jv_k| \leq |v_kv_i|$, and let $n_{i,j}$ denote the number of pieces in the arc $\overset{\frown}{v_iv_j}$ not containing the point v_k, and similarly let $n_{j,k}$ and $n_{k,i}$ denote the number of pieces in the arc $\overset{\frown}{v_jv_k}$ not containing v_i and the number of pieces in the arc $\overset{\frown}{v_kv_i}$ which does not contain the point v_j.
Consider the mapping $\triangle v_iv_jv_k \mapsto (n_{i,j}, n_{j,k}, n_{k,i})$; notice that if i, j, k are so chosen that the conditions $0 < |v_iv_j| \leq |v_jv_k| \leq |v_kv_i|$ are satisfied then $n_{k,i} \geq n_{j,k} \geq n_{i,j} \geq 1$, and also $n_{i,j} + n_{j,k} + n_{k,i} = 9$, whence $1 \leq n_{i,j} \leq n_{j,k} \leq n_{k,i} \leq 7$ for any such triple (i, j, k).
Furthermore, notice that for any two triangles $\triangle v_av_bv_c$ and $\triangle v_iv_jv_k$ with $0 < |v_av_b| \leq |v_bv_c| \leq |v_cv_a|$ and $0 < |v_iv_j| \leq |v_jv_k| \leq |v_kv_i|$ one has

$$(n_{a,b}, n_{b,c}, n_{c,a}) = (n_{i,j}, n_{j,k}, n_{k,i}) \iff \triangle v_av_bv_c \cong \triangle v_iv_jv_k,$$

where $\cong$ denotes that the two triangles are congruent, thus we obtain a bijection between the class of congruent triangles formed by the vertices of a regular 9 gon and the set $\{(\alpha, \beta, \gamma) \in \mathbf{N}^3 : \alpha \leq \beta \leq \gamma \quad \text{and} \quad \alpha + \beta + \gamma = 9\}$; it is easy to list all such triples which are: $(1,1,7)$, $(1,2,6)$, $(1,3,5)$, $(1,4,4)$, $(2,2,5)$, $(2,3,4)$, and $(3,3,3)$; in particular this shows that there are 7 classes of congruent triangles, and since we have already established the existence of at least 10 red triangles, so by the pigeonhole principle it follows that there must be at least two congruent red triangles.

□

1.6 Recursion

Combinatorial problems often revolve around finding the size of a finite set A. Suppose we have a sequence $(A_n)_{n=1}^{\infty}$ of finite sets, and we wish to count the number of elements in these sets, thus letting $(N_n)_{n=1}^{\infty}$ denote the respective sizes of these sets, we wish to find N_n in terms of some parameters which determine the sets A_n. Often these sets A_n are recusrisively defined in the sense that given any $n > m$, for some integer m, the set A_n can be uniquely determined by the sets $A_1, ..., A_{n-1}$, and in such situations it is usually helpful to express the quantities N_n for $n > m$, in terms of $N_1, ..., N_{n-1}$.

Example 1.6.1 (Recursive counting, preliminary example)

Let A_n denote the set of all possible outcomes of distributing n balls among 20 children. Then, $N_n = |A_n|$ represents the number of ways to distribute n distinct balls among 20 children.
Then we all know that $N_n = 20^n$, since each of the n balls has 20 different children as choices to get handed to.
There is another way of finding this quantity N_n. Notice that, if there are $n+1$ balls then, after the first ball is given to some child the remaining n balls can be distributed in exactly N_n many ways, and since the first ball can be given to any one of the 20 students, thus one has $N_{n+1} = 20N_n$. Furthermore a similar argument also suggests that $N_1 = 20$. Putting all this together yields $N_n = 20^n$ for all positive integers n.

Notice that the second approach via recursion somehow masks the details of the underlying procedure, all we needed was finding how the value of the count changes when n changes to $n+1$, and an initial value which in this case was N_1. This is the crux of the recursive method. The following example sheds more light on the recursive strategy.

Example 1.6.2

For a emergency secret mission n agents from n distinct ranks $r_1, ..., r_n$ decide to meet. They are supposed to enter a room one by one in increasing order of their ranks, where n chairs labelled as $c_1, ..., c_n$ are already kept. The agents choose their seats according to the following rule:

- the first person chooses one of the n chairs arbitrarily and occupies that,
- for $i \in \{2, ..., n\}$ the i-th agent after entering the hall checks if the chair labelled as c_i is vacant or not; if it is vacant then he/she occupies it, otherwise he/she occupies one of the vacant seats arbitrarily and occupies it.

Find the total number of possible seating arrangements for these n agents.

Solution. Let N_n denote the number of sitting arrangements possible for these n agents. Then, observe that if the first person occupies c_j then there is only one sitting arrangement; otherwise if he occupies c_j then the agents of ranks $r_2, ..., r_{j-1}$ sit in their own chair and the remaining $n-j+1$ have exactly N_{n-j+1} sitting arrangements.

Thus one has: $N_n = N_1 + N_2 + ... + N_{n-1} + 1$ where the $+1$ comes due to inclusion of the case when the agent of rank r_1 occupies chair c_1. Thus, $N_n = N_{n-1} + (N_1 + ... + N_{n-2} + 1) = 2N_{n-1}$. Furthermore, it is easy to find that $N_1 = 1$, and so putting these together, one has $N_n = 2^{n-1}$ for each $n \in \mathbf{N}$. □

Interested readers are suggested to try solving the above problem without using any recursive strategy.

1.6.1 Fibonacci Counting: a detailed example

We begin with an interesting typical example of Fibonacci counting.

Example 1.6.3

Let n be a positive integer. Suppose that we toss a fair coin n times and record the (ordered) sequence of outcomes. What is the probability of not observing two consecutive heads in the sequence of outcomes ?

We shall not, attempt a complete solution to the question in the sense that we shall not derive a closed form of the probability we are asked to compute. We shall instead derive a recursive expression for the same.

A sketch (deriving the recursion). Let the probability we are looking for be $= p_n$.
Then, $p_n \cdot 2^n$ is the number of ways we can write a sequence of n characters, where each character $\in \{0, 1\}$ such that no two 1 appear as consecutive terms of the sequence.
(Here we pretend 0 to denote Tails and 1 to denote Heads as the outcome of the respective coin toss.)
Let us denote by T_n this number. So, now finding T_n is good enough to serve our purpose since, p_n can simply be computed as $p_n = \frac{T_n}{2^n}$.
Consider the first term of such a sequence of H, T with no two consecutive 1. Then, it is either a 1 or a 0 each with equal probability.
If it is 1 then notice that the second term cannot be 1 and the remaining sub sequence of length $n-2$ cannot have any two consecutive 1. And there are T_{n-2} many such sequences.
And if it is 0 then notice that the only restriction on the sequence is that the remaining sub sequence of length $n-1$ cannot have any two consecutive 1. And there are T_{n-1} many such sequences.
This yields,

$$T_n = T_{n-1} + T_{n-2}.$$

This recursive equation can be framed as

$$p_n = \frac{p_{n-1}}{2} + \frac{p_{n-2}}{4},$$

which gives us a recursive equation for the sequence $\{p_n\}_{n=1}^{\infty}$ of probabilities.

□

Let us record a couple of definitions before proceeding into deeper details.

Definition 1.6.4.

- A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is said to obey the ***Fibonacci recursion*** if $x_{n+2} = x_n + x_{n+1}\ \forall n \in \mathbb{N}$.

- And the sequence $\{F_n\}_{n=1}^{\infty}$ of real numbers defined by $F_1 = F_2 = 1$ and $F_n + F_{n+1} = F_{n+2}$ $\forall n \in \mathbb{N}$ is called the ***Fibonacci sequence***.

Both are named after the infamous Italian mathematician Leonardo of Pisa, later known as Fibonacci. They appear to have first arisen as early as 200 BC in work by Pingala on enumerating possible patterns of poetry formed from syllables of two lengths. In his 1202 book Liber Abaci, Fibonacci introduced the sequence to Western European mathematics, although the sequence had been described earlier in Indian mathematics. A modern translation of Fibonacci's Liber Abaci in the English language can be found in [27].

✍ **Exercise 1.6.5.** Express the sequence $\{T_n\}_{n=1}^{\infty}$ of the number of binary sequences of length n with no block of consecutive k many 1 s in terms of a recursive equation.

In a later section we shall learn about *generating functions* which shall serve as a general method to find solutions to some particular family of recursively defined sequences. And in yet another chapter, we shall discuss about Fibonacci numbers in more details.

1.7 An encounter with generating functions

In a previous section in this chapter we discussed about the Fibonacci recurrence. In this section we present a more elaborate discussion on recurrence equations and the powerful tool of generating functions to solve them.
Given a sequence $(a_n)_{n\geq 0}$ we define the associated ***generating function*** as

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots\dots\dots,$$

which is defined and converges in its disk of convergence.
As we saw in a previous section, often combinatorial problems involve obtaining a certain count which can be sometimes difficult to obtain, but for which one can obtain a recurrence equation in the spirit of Example 1.6.3; once we obtain such a recursion we can use the corresponding generating function for obtaining better insights of the sequence. We begin with an example on how generating functions can help us to find the terms of a sequence which defined through some recursion.

Example 1.7.1

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers satisfying the initial conditions $a_0 = 1$, and $a_1 = 1$, and the recurrence $4a_{n+1} = 4a_n - 4a_{n+1}$ for all positive integers n. Find a formula for a_n in terms of n.

Solution. Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \dots\dots$ be the generating function for the sequence $(a_n)_{n=0}^{\infty}$. Then plugging the values of a_0, a_1 as given we have

$$f(x) - 1 - x = a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots\dots,$$

whence by plugging the given recursive equation one obtains

$$f(x) - 1 - x = (4a_1 - 4a_0)x^2 + (4a_2 - 4a_1)x^3 + \dots\dots\dots$$

$$\implies f(x) - 1 - x = (4a_1 x^2 + 4a_2 x^3 + \dots\dots) - (4a_0 x^2 + 4a_1 x^3 + \dots\dots),$$

and thus we obtain the functional equation

$$f(x) - 1 - x = 4x(f(x) - 1) - 4x^2 f(x).$$

Taking $|x| < \frac{1}{2}$, and solving for $f(x)$ one obtains

$$f(x) = \frac{1 - 3x}{(1 - 2x)^2},$$

and this gives us

$$f(x) = \frac{1}{1 - 2x} - \frac{x}{(1 - 2x)^2} = \sum_{n=0}^{\infty} (2x)^n - x \sum_{n=0}^{\infty} n(2x)^{n-1} = \sum_{n=0}^{\infty} (2^n - n2^{n-1})x^n,$$

and so comparing the two power series expansions of $f(x)$ one obtains (since equality of two power series in a non-degenerate interval implies equality of the coefficients)

$$a_n = 2^n - n2^{n-1}.$$

Checking for consistency of the initial values we see that $a_0 = 1, a_1 = 1$ are indeed true. □

This is the broad picture of how generating functions are used to solve recurrence equations.

✍ **Exercise 1.7.2.** Using generating functions solve the Fibonacci recursion defined by $F_1 = 1 = F_2$, and $F_{n+2} = F_{n+1} + F_n$ for each positive integer n.

Often operations on sequences of coefficients can be translated to operations on their generating functions.

✍ **Exercise 1.7.3.** If $(a_n)_{n\geq 0}$, and $(b_n)_{n\geq 0}$ are two sequences of real numbers, and if $A(t), B(t)$ be their generating functions respectively, then find the generating function for the sequence $(c_n)_{n\geq 0}$, where we define $c_n := a_n \cdot b_n$ to be the product of a_n, and b_n for each $n \geq 0$.

In some situations it may be more convenient to work with the the series $x^{a_0} + x^{a_1} + x^{a_2} + \ldots..$ instead of the usual generating function.

Example 1.7.4 (1998 IMO Shortlist)

Let $(a_n)_{n\geq 0}$ be an increasing sequence of non-negative integers such that every non-negative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j, k are not necessarily distinct. Determine the value of a_{1998}.

Solution. For $|x| < 1$, let $f(x) = \sum_{i=0}^{\infty} x^{a_i}$. Then the given condition is equivalent to saying

$$f(x)f(x^2)f(x^4) = \sum_{n\geq 0} x^n.$$

Now from the geometric series formula we know that $\sum_{n\geq 0} x^n = \frac{1}{1-x}$, thus we get

$$f(x)f(x^2)f(x^4) = \frac{1}{1 - x}.$$

Now replacing x by x^2 we get

$$f(x^2)f(x^4)f(x^8) = \frac{1}{1-x^2}.$$

From these two equations we get $f(x) = (1+x)f(x^8)$, and thus repeating this recursively gives us

$$f(x) = (1+x)\cdot(1+x^8)\cdot(1+x^{8^2})\cdot\ldots\ldots\ldots \quad .$$

Now expanding the right hand side of the above equation we see that $a_0, a_1, \ldots..$ are precisely those non negative integers whose base 8 representation has the digits 0 and 1 only.
Now writing 1998 as $1998 = 2 + 2^2 + 2^3 + 2^6 + 2^7 + 2^8 2^9 + 2^{10}$, we calculate

$$a_{1998} = 8 + 8^2 + 8^3 + 8^6 + 8^7 + 8^8 + 8^9 + 8^{10} = 1227096648.$$

□

For a broader text on generating functions one may read [30].

1.8 Algebraic techniques in combinatorics

In this section, we shall demonstrate some examples of how algebraic techniques can be used to deal with combinatorial problems.

Linear algebraic tools in combinatorics

We begin our discussion with a demonstration of how linear algebraic methods can be used to solve combinatorial problems; we begin with the following well-known example.

Example 1.8.1 (St. Petersburg)

There are n students in a class. On one occasion they go for ice cream in groups of size at least 2. After $k > 1$ groups have gone, it was noticed that every two distinct students have gone together (in the same group) exactly once. Prove that $n \leq k$.

Solution.
First notice that if there was a student $\mathcal{S}$ who went only once for ice cream, then all the other students must have gone with him together. So, this implies $k = 1$, which contradicts the given condition that $k > 1$.
So it means that every student went ≥ 2 times.
Next suppose that we label the students as $s_1, s_2, \ldots, s_n$, and we consider the vectors $\vec{v_1}, \vec{v_2}, \ldots.., \vec{v_n} \in \mathbb{R}^k$, where the i - th co-ordinate of $\vec{v_t}$ is equal to 1 if the t -th student went in the i -th group.
Then from the given condition that each un-ordered pair of students went together exactly once we get, $\vec{v_i}\cdot\vec{v_j} = 1$ whenever $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.
Also notice that the condition (which we just proved) that each student went ≥ 2 times means that $|\vec{v_i}|^2 \geq 2$ for each $i \in \{1, 2, \ldots, n\}$.
Now if $n > k$, then it means that the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is not linearly independent[3].

[3] this is where we use the (well known) algebraic result : *no subset of $\mathbb{R}^k$ having more than k elements in it is linearly independent.*

Hence there exists $\vec{w} = (w_1, w_2, ..., w_n) \neq \vec{0}$ such that, $\sum_{i=1}^{n} w_i\vec{v_i} = \vec{0}$, and

$$\implies \sum_{i=1}^{n}(w_i\vec{v_i}) \cdot \sum_{i=1}^{n}(w_i\vec{v_i}) = 0$$

$$\implies \sum_{i=1}^{n} w_i^2(|\vec{v_i}|^2 - 1) + \left(\sum_{i=1}^{n} w_i\right)^2 = 0$$

But from the conditions we have on the vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ and the assumptions on w_i, it can be easily seen that the equation above cannot be true.
This refutes our assumption, and hence completes the solution.

$\square$

Let us see yet another example, which happens to be very popular for the purpose of illustrating algebraic methods in combinatorics.

Example 1.8.2 (Odd town with even intersections)

A town with n inhabitants has m clubs such that each club has an odd number of members and any two different clubs have an even number of common members. Prove that $m \leq n$.

Solution. Consider the n dimensional vector space $\mathbb{Z}_2^n$ of n tuples of 0 and 1 over the field $\mathbb{Z}_2 = \{0, 1\}$ under the modulo 2 operations. Also consider enumerating the residents of the town.
We represent each club $\mathcal{C}$ by its incidence vector $\mathbf{1}_{\mathcal{C}}$ where the i th resident of the town is a member of club $\mathcal{C}$. We claim that these vectors are linearly independent. Suppose one has $\vec{z} = \sum_{\mathcal{A} \text{ is a club}} t_{\mathcal{A}}\mathbf{1}_{\mathcal{A}} = \vec{0}$ for some coefficients $t_{\mathcal{A}}$ s in $\mathbb{Z}_2$; to show that the incidence vectors are linearly independent it is necessary and sufficient to show that all the coefficients $t_{\mathcal{A}}$ must be 0.
Fix any arbitrary club $\mathcal{B}$ in the town, and consider the inner product equation

$$\vec{z} \cdot \mathbf{1}_{\mathcal{B}} = 0.$$

Now by the mentioned odd town rules one must have

$$0 = \vec{z} \cdot \mathbf{1}_{\mathcal{B}} = \sum_{\mathcal{A} \text{ is a club}} t_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}} \cdot \mathbf{1}_{\mathcal{B}}) = t_{\mathcal{B}},$$

where all the operations are done over $\mathbb{Z}_2$. Thus we conclude that $t_{\mathcal{B}} = 0$, and since this club $\mathcal{B}$ was chosen arbitrarily, so it follows that $t_{\mathcal{C}} = 0$ for each club $\mathcal{C}$ in the town, thus the claim follows.

$\square$

Following is the dual version of the odd town problem which can be solved in a similar way.

✍ **Exercise 1.8.3** (Even town with odd intersections)**.** A town with n inhabitants has m clubs such that each club has an even number of members and any two different clubs have an odd number of common members. Prove that $m \leq n$.

A related variant of the odd town with even intersections problem is the even town with even intersections problem(see https://scoutmathematics.wordpress.com/2021/04/11/odd-even-town-with-odd-even-intersections/). Its solution however demands familiarity with some more familiarity with vector spaces (especially about orthogonal complements of a vector subspace and their dimensions).

Example 1.8.4 (Even town with even intersections)

If S is a finite set having n elements in it and if $\mathcal{S}$ is a set of subsets of S such that for any two sets $A, B \in \mathcal{S}$ one has $|A \cap B|$ is even, then prove that: $|\mathcal{S}| \leq 2^{\lfloor n/2 \rfloor}$; furthermore this bound can be attained.

Solution. First of all notice that if we consider $\lfloor n/2 \rfloor$ many disjoint (unordered) pairs of elements of S then taking any arbitrary union of these give us a set $\mathcal{S}$ of subsets of S satisfying the desired properties and with $|\mathcal{S}| = 2^{\lfloor n/2 \rfloor}$.
Thus it suffices to show that if $\mathcal{S}$ is a set of subsets of S satisfying the mentioned properties then $|\mathcal{S}| \leq 2^{\lfloor n/2 \rfloor}$. To this end, we identify each element of $\mathcal{S}$ with its incidence vector whose coordinates are the indicators of whether the corresponding element is in the set S or not. Then it suffices to show that this set of vectors has $\leq 2^{\lfloor n/2 \rfloor}$ many elements in it, thus in turn it suffices to show that this set of vectors spans a subspace V of $\mathbb{Z}_2^n$ of dimension $\leq \lfloor n/2 \rfloor$. For doing this, we notice that if K is the set of such incidence vectors then by the given property it follows that $\forall a, b \in K$ one has $a \cdot b = 0$, thus if V denotes the subspace spanned by K then (by linearity of dot product) $a \cdot b = 0$ holds true for any $a, b \in V$. Let $V^{\perp} := \{u \in \mathbb{Z}_2^n : u \cdot v = 0 \quad \forall v \in V\}$; notice that $V^{\perp}$ is a subspace of $\mathbb{Z}_2^n$ (often called the orthogonal complement of V in $\mathbb{Z}_2^n$), and furthermore we have by the assumed condition on K (and hence on its span) $V \subseteq V^{\perp}$, thus $\mathsf{dim}(V) \leq \mathsf{dim}(V^{\perp})$; also since V is a subspace of $\mathbb{Z}_2$, so one has $\mathsf{dim}(V) + \mathsf{dim}(V^{\perp}) = n$, thus $\mathsf{dim}(V) \leq \lfloor n/2 \rfloor$ and so $2^{\lfloor n/2 \rfloor} \geq |V| \geq |K|$, and the claim follows.

□

There is a similar dual to this problem which we leave as an exercise for the reader:

✍ **Exercise 1.8.5** (Odd towns with odd intersections)**.** If S is a finite set having n elements in it and if $\mathcal{S}$ is a set of subsets of S such that for any two sets $A, B \in \mathcal{S}$ one has $|A \cap B|$ is odd, then find an achievable upper bound for $|\mathcal{S}|$.

Partially ordered sets

An ***ordering*** on a set S refers to a subset of $S \times S$.

Definition 1.8.6. A set $(\mathcal{M}, \preceq)$ under an ordering $\preceq$ is called a ***partially ordered set*** (abbreviated to ***poset***), if

i. $a \preceq a$, for every $a \in \mathcal{M}$;

ii. $a \preceq b \;\&\; b \preceq c \implies a \preceq c$, for any $(a, b, c) \in \mathcal{M}^3$;

iii. $a \preceq b \;\&\; b \preceq a \implies a = b$, for any $(a, b) \in \mathcal{M}^2$.

If $(\mathcal{M}, \preceq)$ is a partially ordered set, then $\preceq$ is said to be a ***partial ordering*** on $\mathcal{M}$.

Notations: For a poset $(\mathcal{M}, \preceq)$ and $(a, b) \in \mathcal{M}^2$, $a \succeq b$ means the same as $b \preceq a$, $a \prec b$ would mean $a \preceq b \;\&\; a \neq b$, and $a \succ b$ would mean $b \prec a$.
Notice that a partial ordering on a set has the reflexive, the anti-symmetric and the transitive properties.
Also, notice that there may be pairs of distinct elements in a poset which can't be compared.

Example 1.8.7

Some standard examples of posets may include :

i. The set of real numbers $\mathbf{R}$ forms a partially ordered set under the usual inequality comparison among real numbers.

ii. For a set S, and a (non-empty) set $\mathcal{F}$ of subsets of S, the inclusion ordering $(a \preceq b \iff a \subseteq b)$ is a partial ordering on $\mathcal{F}$.

iii. Consider a set S, and consider a (non-empty) subset $\mathcal{F}$ of the set of all partitions of S. On $\mathcal{F}$ we define $\preceq$ by the rule : $a \preceq b$ if any block[a] of a is contained in a block of b.

[a]for a partition $\mathcal{E} = \{ e_\alpha \ : \ \alpha \in \text{some index set } \mathcal{I} \}$, the e_α s are known as ***blocks*** of $\mathcal{E}$ or ***parts*** of the partition $\mathcal{E}$

✍ **Exercise 1.8.8.** Check that the examples we have mentioned are indeed examples of posets.

Notation and terminology : In a poset $\mathcal{P} = (\mathcal{M}, \leq)$, we write $a \lessdot b$, if $a \leq b$, $a \neq b$, and there is no c with $a < c < b$. If $a \lessdot b$, then we say b ***covers*** a, or, we also say a ***is covered by*** b.

Definition 1.8.9. A poset $(\mathcal{M}, \preceq)$ in which for any two elements a, b of $\mathcal{M}$ we have either $a \preceq b$ or $b \preceq a$, is called a ***totally ordered set*** or ***linearly ordered set***. An for a totally ordered set $(\mathcal{M}, \preceq)$, the ordering $\preceq$ is called a ***total ordering*** or ***linear ordering*** on $\mathcal{M}$.

Definition 1.8.10. A linearly ordered set $\mathcal{W} = (\mathcal{A}, \preceq)$ is called a ***well-ordered set*** if any non-empty subset of elements of $\mathcal{W}$, has a least element.
And if $\mathcal{W} = (\mathcal{A}, \preceq)$ is a well-ordered set, then we call $\preceq$ to be a ***well-ordering*** on $\mathcal{A}$.

Example 1.8.11

The set of natural numbers under the usual comparison ordering $(\mathbf{N}, \leq)$ is a well-ordered set and hence also a linearly ordered set.
However, $(\mathbf{R}, \leq)$ is a linearly ordered set, but not a well-ordered set.

Definition 1.8.12. If $(\mathcal{W}, \preceq)$ is a well-ordered set, and $w \in \mathcal{W}$, then the set $\mathcal{W}_{<w} := \{x \in \mathcal{W} \ : \ x < w\}$ is called ***initial segment*** of $\mathcal{W}$ given by w.

✍ **Exercise 1.8.13.** Let $(\mathcal{W}, \preceq)$ be a well-ordered set, and let $f : \mathcal{W} \to \mathcal{W}$ be an increasing function. Show that $w \preceq f(w)$ is true $\forall\ w \in \mathcal{W}$.

Definition 1.8.14. In a poset $(\mathcal{M}, \preceq)$, for any $(x, y) \in \mathcal{M}^2$, if $x \preceq y$ or $y \preceq x$ then, we say the elements x, y are ***comparable*** in the poset $(\mathcal{M}, \preceq)$, and otherwise, we say that x and y are ***not comparable***.

Thus in a linearly ordered set any two elements should be comparable among themselves.

Definition 1.8.15. If $(\mathcal{M}, \preceq)$ is a poset, then a subset $\mathcal{C} \subseteq \mathcal{M}$ is called a ***chain***, if any two elements in $\mathcal{C}$ are comparable (that is to say, if for any $a \in \mathcal{C}$, & any $b \in \mathcal{C}$, we have either $a \preceq b$, or $b \preceq a$).
Thus, chains of a poset are the linearly ordered subsets of the poset.

Notice that any finite chain of a poset $(\mathcal{A}, \preceq)$ can be enumerated as $a_1 \preceq a_2 \preceq ... \preceq a_m$ where $(\{a_1, a_2, ..., a_m\} \subseteq \mathcal{A})$.

Definition 1.8.16. If $(\mathcal{M}, \preceq)$ is a poset, then a subset $\mathcal{A} \subseteq \mathcal{M}$ is called an ***anti-chain***, if for any two $a, b \in \mathcal{A}$ we have $a \preceq b \iff a = b$.
In other words, no two distinct elements of an anti-chain are comparable.

Thus, chains are strings of pairwise comparable elements, whereas in anti-chains any two elements are non comparable.

⚠ **Word of caution 1.8.17.** Be careful to avoid confusion between anti-chains of a poset and subsets of a poset which are *not* chains. For example in the poset $(\{\{1\}, \{1,2\}, \{3\}, \{2\}\}, \subseteq)$, (under the inclusion ordering), the set $\{\{1\}, \{1,2\}, \{3\}\}$ is not a chain, but it is also not an anti-chain.

Exercise 1.8.18. If $\mathcal{A}$ is an infinite set, and $\preceq$ is a partial ordering on $\mathcal{A}$, then show that the poset $(\mathcal{A}, \preceq)$ will either have an infinite chain, or an infinite anti-chain (or, both). (Hint: consider the maximal chains in the poset $(\mathcal{A}, \preceq)$.)

Exercise 1.8.18 is a particular instance of a more general result namely the famous Erdős-Szekeres theorem.

Three grand theorems: Dilworth, Mirsky and Erdős-Szkeres

We discuss three interesting results, which have relevance to posets and are of heavy importance in algebraic combinatorics.

Theorem 1.8.19 (Dilworth's theorem)

If $\mathcal{P} = (\mathcal{M}, \preceq)$ is a finite poset and $\mathcal{A}$ is a longest anti-chain in this poset, then there exists a decomposition of $\mathcal{P}$ into $|\mathcal{A}|$ many disjoint distinct chains.

Proof. We shall induce on the number of elements in $\mathcal{M}$. We notice that if $|\mathcal{M}| = 1$, then the result holds trivially, since in that case any subset of $\mathcal{M}$ is either empty or forms both a chain and an anti-chain under the ordering $\preceq$.
So, now let $\ell \in \mathbf{N}$ be a positive integer such that $\forall k \in \mathbf{N}, \& k \leq \ell$ the assertion made in Dilworth's theorem is true, when the poset has k many elements. And let, now the poset have $\ell + 1 = n$ many elements in it.
To begin with, we notice that if $\mathcal{P}$ itself is an anti-chain then the result holds true evidently. So, let us consider the case when $\exists$ at least two distinct elements in the poset which are comparable. Now, let α be a minimal element of $\mathcal{P}$, β be a maximal element of $\mathcal{P}$, such that $\alpha \preceq \beta$, & $\alpha \neq \beta$. Let $\mathcal{W} = \mathcal{M} \setminus \{\alpha, \beta\}$. Then, $\mathcal{W}$ has size $= n - 2$. Now, we notice that, if $\{\alpha, \beta\} \cap \mathcal{A} \neq \phi$, then we can use the induction hypothesis and merge chains appropriately to obtain a chain decomposition of the poset having the desired property.
Otherwise, we define $\mathcal{P}_{\succeq} := (\{x \in \mathcal{M} \ : \ \exists a \in \mathcal{A} \text{ such that } a \preceq x\}, \preceq)$, and $\mathcal{P}_{\preceq} := (\{x \in \mathcal{M} \ : \ \exists a \in \mathcal{A} \text{ such that } x \preceq a\}, \preceq)$. Then, we see that, since $\mathcal{A}$ is an anti-chain, so $\mathcal{P}_{\succeq} \cap \mathcal{P}_{\preceq} = \mathcal{A}$, and due to maximality of $\mathcal{A}$, we get $\mathcal{P}_{\succeq} \cup \mathcal{P}_{\preceq} = \mathcal{P}$. Now, we

see that $\alpha \notin \mathcal{P}_{\succeq}$, & hence $\alpha \in \mathcal{P}_{\preceq}$ and $\beta \notin \mathcal{P}_{\preceq}$, & hence $\beta \in \mathcal{P}_{\succeq}$. This is because, $\{\alpha, \beta\} \cap \mathcal{A} = \phi$, and α is a minimal element & β is a maximal element, and $\mathcal{P}_{\succeq} \cup \mathcal{P}_{\preceq} = \mathcal{P}$. Thus, we have $|\mathcal{P}_{\succeq}| \leq n-1$, and $|\mathcal{P}_{\preceq}| \leq n-1$.
And hence by the induction hypothesis, it follows that $\mathcal{P}_{\succeq}$ can be written as a union of m chains, and $\mathcal{P}_{\preceq}$ can be written as a union of m chains.
Now, we bunch chains from the decomposition corresponding to $\mathcal{P}_{\succeq}$ and $\mathcal{P}_{\preceq}$ pairwise as described below :

- we take a chain from the chain decomposition of $\mathcal{P}_{\succeq}$ and one chain from the chain decomposition of $\mathcal{P}_{\preceq}$ and bunch them by placing one after the other ;
- we continue this process where at each step we choose one chain from each of the two chain decomposition which have not yet been chosen by us in some previous step, and then bunch them by placing one after the other.

This gives us a chain decomposition of the poset $\mathcal{P}$ as union of m chains.
This completes the proof of Dilworth's theorem. □

A dual version of Dilworth's theorem, popularly called Mirsky's theorem asserts the following :

Theorem 1.8.20 (Mirsky's theorem)
If $\mathcal{P} = (\mathcal{M}, \preceq)$ is a finite poset having a maximal chain $\mathcal{C}$, then this poset can be written as a union of $|\mathcal{C}|$ many anti-chains.

Exercise 1.8.21. Prove Mirsky's theorem.

An interesting consequence of Dilworth's theorem is Erdős-Szekeres theorem. Below we mention the theorem alongwith an alternate proof of it. The interested reader is suggested to try proving Erdős-Szekeres theorem using Dilworth's theorem to get a hands-on experience of using Dilworth's theorem.

Theorem 1.8.22 (Erdős-Szekeres theorem)
If $(\mathcal{A}, \preceq)$ is a finite poset with $|\mathcal{A}| > mn$ for some positive integers m, n, then there exists either a chain of length $\geq m+1$ in this poset or this poset has an anti-chain of length $\geq n+1$.

Alternate proof of the theorem. Let $\mathcal{A} = \{a_1, a_2, ..., a_u\}$, (where $u = |\mathcal{A}| > mn$), be an enumeration[4] of the elements of $\mathcal{A}$, such that such that for any $g \in \{1, 2, ..., u\}$, and any $j \in \{1, 2, ..., u\}$, we have :

$$i \leq j \ \ \& \ \ a_j \preceq a_i \quad \Longrightarrow \quad i = j.$$

For any $i \in \{1, 2, ..., u\}$, let m_i be the length of the longest chain with first element $= a_i$, and let n_i be the length of the longest anti-chain with first element $= a_i$.
Then, first notice that $i \neq j \implies (m_i, n_i) \neq (n_i, n_j)$. This is because, for any $i \in \{1, 2, ..., u\}$, and any $j \in \{1, 2, ..., u\}$, and $i < j$, if a_i, a_j are comparable then we must have $a_i \prec a_j$, and then, $m_i \geq m_j + 1$. While on the other hand, if a_i, a_j are not comparable, then $n_i \geq n_j + 1$. Thus,

$$|\{(m_g, n_g) \ : \ g \in \{1, 2, ..., u\}\}| = u \geq mn + 1.$$

[4] the fact that $\mathcal{A}$ is a finite set, implies existence of such an enumeration with the mentioned property

Therefore, either

$$|\{m_g \ : \ g \in \{1, 2, ..., u\}\}| \geq m + 1, \quad \text{or,}$$
$$|\{n_g \ : \ g \in \{1, 2, ..., u\}\}| \geq n + 1.$$

Which, means either $(\mathcal{A}, \preceq)$ contains a chain of length $\geq m + 1$, or $(\mathcal{A}, \preceq)$ contains an anti-chain of length $\geq n + 1$. □

Below we demonstrate an interesting application of Mirsky's theorem :

Example 1.8.23 (Iran 2006)

Let $k \in \mathbf{N}$. Let S be a finite set of bounded intervals in $\mathbf{R}$, such that for any collection of $k + 1$, many intervals in S, there exist at least two with a non-empty intersection. Prove that there exists a set $\mathcal{A}$, of k many points in $\mathbf{R}$, which intersects every interval in S.

Solution. Let $|S| = n$. We consider enumerating S as $S = \{I_1, I_2, ..., I_n\}$. We define a partial order $\preceq$ on $\{1, 2, ..., n\}$, by setting : $i \preceq j$ if and only if

- $i = j$ or
- $\max(I_i) < \min(I_j)$.

It is easy to verify that $\preceq$ is indeed a partial ordering. Let $\mathcal{P} = (\{1, 2, ..., n\}, \preceq)$ be this poset.
Notice that, a chain in this poset, corresponds to pairwise disjoint intervals in S. Also, an anti-chain in this poset corresponds to pairwise intersecting intervals in S.
Now, notice that,

$(*)$ if A, B, C are three sub-intervals of $\mathbf{R}$, with $A \cap B, B \cap C$ and $C \cap A$ non-empty, then $A \cap B \cap C$ must also be non-empty.

Now, getting back to our problem, we notice that the given conditions in the problem imply, the longest chain in $\mathcal{P}$ must have size $\leq k$, and hence by Mirsky's theorem it follows that $\mathcal{P}$ can be written as a union of $\leq k$ many anti-chains. Let $\mathcal{M}$ be a set of $\leq k$ many anti-chains whose union is $\mathcal{P}$.
Also from $(*)$ it follows that, $A \subset \{1, 2, ..., n\}$ is an anti-chain in $\mathcal{P}$, if and only if

$$\bigcap_{a \in A} I_a \neq \phi.$$

So, now, we choose one element from the intersection of intervals corresponding to each anti-chain $\in \mathcal{M}$.
This gives us a set $\mathcal{A}_0$ of $\leq k$ many elements which intersect every interval in S. Now, we can add points to $\mathcal{A}_0$, (if required : if $|\mathcal{A}_0| < k$) to get a set $\mathcal{A}$ of k many points on the real line which intersect every interval in S.
This completes the solution.

□

Below are some interesting exercises, which the readers are suggested to try solving for getting a hands-on experience of applying the theorems we discussed in this section.

Exercise 1.8.24 (Hall's theorem on matching in bipartite graphs)**.** Let G be a finite, simple, bipartite graph with vertex bi-partitions $\mathcal{A}, \mathcal{B}$. Prove that G contains a complete matching from $\mathcal{A}$ to $\mathcal{B}$, if and only if for every $S \subseteq \mathcal{A}$, we have $|N_G(S)| \geq |S|$.
(Here, $N_G(S)$ denotes the set of all and only those nodes of G, which are adjacent to some node $\in S$, in the graph G.)

1.9 A few words about the probabilistic method

Often probability can be used as a tool for a variety of purposes in combinatorics. One of the classic examples is the use of probability to derive a lower bound on the (n, n) -th Ramsey number $R(n, n)$. Here we shall just illustrate a few instances where the probabilistic method comes handy, but the scope of the probabilistic method extends to much more. For more details one can refer to [3]. A basic fact which can be used in various modified or unmodified forms in combinatorics is

Lemma 1.9.1

Let Ω be a finite set and w_1, w_2 be two fixed non empty subsets of Ω. Suppose X is a random variable with $P(X \in \Omega) = 1$ and such that $P(X \in w_1) + P(X \in w_2) > 1$. Then $w_1 \cap w_2 \neq \phi$ (the empty set).

Proof. This is a very straightforward and elementary result and can be proved directly from the Law of Total Probability in the way : if on the contrary, we had $w_1 \cap w_2 = \phi$, then $w_2 \subseteq \Omega \setminus w_1$, and hence
$1 < P(X \in w_1) + P(X \in w_2) \leq P(X \in w_1) + P(X \in \Omega \setminus w_1) = P(X \in \Omega) = 1$, thus yielding a contradiction. This shows that our assumption was false. Hence the negation of our assumption must be true. This completes the proof. $\square$

Even if this result seems to be very elementary, this has widespread applications in combinatorics. Before we move to discussion about the big example of deriving a lower bound on Ramsey numbers, let us illustrate a relatively easy example (see for instance [5, adityaguharoy]).

Example 1.9.2 (Putnam 2004, Problem number B2)

Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

Solution. We consider the following experiment. Suppose we have a collection of $m + n$ distinct balls, and $m + n$ distinct boxes of which m boxes are of black color and n many boxes are of white color. We consider picking m balls at random (and without replacement) from the $m + n$ balls, painting them black and painting the remaining n balls white.
Then, we consider distributing the balls to the boxes in such a way that black balls get placed only in black boxes and white balls in white boxes ; a box can contain more than one balls and some boxes can be empty. Then, the number of ways in which this task can be done (i.e., the number of different outcomes which one can observe at the end of this task) is clearly $\binom{m+n}{m} \cdot m^m \cdot n^n$.
Now, we consider a slightly different experiment. Once again consider $m + n$ distinct balls and $m + n$ distinct boxes of which m boxes are black and n boxes are white. We randomly allocate the balls to the boxes. Then, the number of ways in which this task can be done (i.e., the number of different outcomes which one can observe at the end of this task) is clearly $(m + n)^{m+n}$.
Now observe that the set of outcomes of the first experiment is a strict subset (since

$m, n > 1$) of the set of outcomes of the second experiment, whence the probability that we end up with an outcome obtained from the first experiment by performing the second experiment is < 1.
Thus, it follows that:

$$\binom{m+n}{m} \cdot m^m \cdot n^n < (m+n)^{m+n},$$

which upon rearranging gives

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

□

Now, we come to the grand example of Ramsey numbers. But first a definition:

Definition 1.9.3. Let m, n be two positive integers. Then the (m, n) th Ramsey number written as $R(m, n)$ is the smallest positive integer N, such that given any arbitrary 2-coloring of the edges of the complete graph K_N on N vertices using two colors say red and blue, it has either a blue clique of size m or a red clique of size n.

Before proceeding the readers are encouraged to solve the following elementary exercise:

✍ **Exercise 1.9.4.** Show that $R(m, n) = R(n, m)$, for all positive integers m, n.

A common topic of combinatorics where the probabilistic method has been used quite extensively is graphs. In fact, even in some other applications of the probabilistic method, arguments are often framed in the vocabulary of graphs. The following theorem provides a good demonstration of the power of probabilistic method in combinatorics.

Theorem 1.9.5 (Lower bound on the (n, n) th Ramsey number, preliminary version)
If m is a positive integer, and $R(m, m)$ denotes the (m, m) th Ramsey number, then we have $R(m, m) \geq 2^{m/2}$.

Proof. Consider any positive integer $n \leq 2^{m/2}$, and construct the complete graph K_n on n vertices. Then we randomly assign to each edge a color which is either red or blue each with equal probabilities and assigning colors to different edges independently. Let $\mathcal{E}_R$ denote the event of finding at least one red clique of size m in this coloured graph, and let $\mathcal{E}_B$ denote the event of finding at least one blue clique of size m in this graph.
In view of the above discussion it suffices to show that $P(\mathcal{E}_R \cup \mathcal{E}_B) < 1$; a verification which is left for the readers.
Now, to this end we use the trivial estimate

$$P(\mathcal{E}_R \cup \mathcal{E}_B) \leq P(\mathcal{E}_R) + P(\mathcal{E}_B) = 2P(\mathcal{E}_R).$$

Now, notice that K_n has $\binom{n}{m}$ many cliques of size m, and let these be numbered as $\mathcal{C}_1, \ldots\ldots, \mathcal{C}_{\binom{n}{m}}$.
Let for any $j \in \{1, \ldots\ldots, \binom{n}{m}\}$, R_j denote the event that $\mathcal{C}_j$ is a red clique and B_j denote the event that $\mathcal{C}_j$ is a blue clique.
Then, we have

$$2P(\mathcal{E}_R) \leq 2 \sum_{j=1}^{\binom{n}{m}} P(R_j)$$

$$= 2\sum_{j=1}^{\binom{n}{m}} \left(\frac{1}{2}\right)^{\binom{m}{2}}$$

$$= 2\binom{n}{m}\left(\frac{1}{2}\right)^{\binom{m}{2}}$$

$$\leq 2\frac{n^m}{m!}\left(\frac{1}{2}\right)^{\binom{m}{2}}$$

$$\leq \frac{2^{1+m/2}}{m!} < 1,$$

thus the claim follows. □

The bound on the right hand side of the above is not tight, and we can improve that as discussed below, using the machinery of Stirling's approximation and by refining some steps of the above proof.

Lemma 1.9.6 (Stirling's approximation)

One has the following approximation for factorial

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

This result is a folklore and can be proved in several ways. For a broader discussion on Stirling's approximation and its proof refer to the appendix on this topic.

Theorem 1.9.7 (Lower bound on the (n, n) -th Ramsey number)

There exists a positive integer N such that for all positive integers $m \geq N$, the inequality

$$R(m, m) \geq \frac{m \cdot 2^{\frac{m}{2}}}{e \cdot \sqrt{2}}$$

is true.

Proof (due to Erdős). Suppose n is a positive integer, and we consider coloring the edges of the complete graph K_n using two colors, say red and blue, where each color is equally likely to get assigned and the colors are assigned independently to the different vertices. We consider fixing a set of m vertices in this colored graph and we consider the subgraph K_m induced by these vertices. Then, the probability that this subgraph is monochromatic is $2^{1-\binom{m}{2}}$.

Now, suppose we do not fix the set of m vertices but instead look over every possible set of m vertices in the colored K_n. Then the number of such subgraphs will be $\binom{n}{m}$, and so by the law of addition one has the trivial bound

$$P(\text{ finding at least one monochromatic } K_m \text{ in this colored graph }) \leq \binom{n}{m} \cdot 2^{1-\binom{m}{2}}.$$

Thus, if $R(m, m) \leq n$, then one must have $\binom{n}{m} \cdot 2^{1-\binom{m}{2}} \geq 1$. Now, let N denote the smallest positive integer satisfying $\binom{N}{m} \cdot 2^{1-\binom{m}{2}} \geq 1$. Then one must have $R(m, m) \geq N$.

So we have

$$R(m,m) \geq N = (N^m)^{1/m} > \left(\binom{N}{m} m!\right)^{1/m} \geq \left(2^{\binom{m}{2}-1} m!\right)^{1/m} = 2^{m/2-1/2-1/m} \cdot (m!)^{1/m}.$$

Now, using Stirling's approximation one obtains that for large m, one has

$$R(m,m) \geq \frac{m \cdot 2^{m/2}}{e\sqrt{2}} \left(\left(\frac{\pi}{2}\right)^{\frac{1}{2m}} m^{\frac{1}{2m}} \right).$$

Since $\left(\frac{\pi}{2}\right)^{\frac{1}{2m}} m^{\frac{1}{2m}} \geq 1$, so one obtains for large m one has the bound

$$R(m,m) \geq \frac{m \cdot 2^{\frac{m}{2}}}{e \cdot \sqrt{2}}.$$

□

Before moving on the next topic, let us discuss yet another beautiful application of probability in solving a question of combinatorial flavour. In this one, we shall however just outline the solution and the reader is requested to fill up the details for getting a hands-on experience of handling probabilistic arguments.

Example 1.9.8

Is it true that for any positive integer n, we can find a n by n matrix $\mathcal{A}_n$, having each entry $\in \mathbf{R}$ which is not invertible, but changing any one entry of $\mathcal{A}_n$ will make it invertible ?

Sketch of solution.
Consider the hyperplane $\mathcal{H} := \{(x_1, x_2, ..., x_n) : x_1 + x_2 + ... + x_n = 0\} \subset \mathbb{R}^n$. Now, pick any n points $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ in $\mathcal{H}$ independently according to some probability distribution of choice (say Gaussian), and stack them as rows of $\mathcal{A}_n$. That is, $\mathbf{X}_i$ will serve as the i th row of $\mathcal{A}_n$.
Then,

- what can we tell about the determinant of $\mathcal{A}_n$, and its minors ?
- does changing any entry of the matrix $\mathcal{A}_n$ also alter its determinant ?

Proceeding this way (answering the two simple questions above), will allow us to make the desired conclusion about the answer of the question, thus completing the solution. □

There are non probabilistic solutions too, but they involve dealing with some untidy expressions of cofactors and all. This elegant probabilistic solution is due to professor Fedor Petrov.

✍ **Exercise 1.9.9.** Complete the above solution.

Another important tool in probabilistic combinatorics is expectation of a random variable. In almost all instances, the random variables we would deal with in this text can be assigned a measure of central behaviour known as ***expectation***. [5]

[5] It should be cautioned that there are random variables without a well-defined notion of expected value, unless one relaxes some of the demands about properties such a measure should possess.

Definition 1.9.10. Suppose X is a random variable taking values in a finite or a countably infinite subset $C := \{c_i : i \in \mathcal{I}\} \subseteq \mathbf{R}$, where $\mathcal{I} \subseteq \mathbf{N}$. Then the ***expected value*** (or ***expectation***) $\mathbf{E}[X]$ of the random variable X is defined to be the quantity

$$\mathbf{E}[X] := \sum_{i \in \mathcal{I}} c_i \mathbf{P}\{X = c_i\},$$

provided that this sum is absolutely convergent.

One can define expected values of random variables in a different way even when the range of the random variable is uncountable, however in this case there are a few technical issues involved; we would mostly deal with situations when C is a finite set, and for such situations the expected value will always exist and will be finite.
For a thorough discussion on expected value of random variables one can read [26].
Following is a very interesting property of expected values which is often quite useful in combinatorics.

Proposition 1.9.11 (Linearity of expectation)

If $X_1, \ldots.., X_n$ are random variables each taking values in a countable set and having finite expectations and if $c_1, \ldots., c_n$ are finite real numbers then

$$\mathbf{E}[c_1 X_1 + \ldots.. + c_n X_n] = c_1 \mathbf{E}[X_1] + \ldots. + c_n \mathbf{E}[X_n].$$

Proof. It is enough to prove the result for $n = 2$ since the result for any general n follows directly from it, and for $n = 2$ the claim can be proved directly from the definition of expectation; details left □

Another thing which often comes handy in combinatorics is the following elementary result.

Proposition 1.9.12

If X is a random variable taking only countably many values, having an expected value, and if w is a real number such that $\mathbf{P}\{X \leq w\} = 1$ then $\mathbf{E}[X]$ exists and $\mathbf{E}[X] \leq w$. Also if $\mathbf{P}\{X < w\} = 1$ then $\mathbf{E}[X] < w$.

Proof. Follows from the definition of expected value. □

An immediate consequence of this is the following interesting result.

Corollary 1.9.13

If X is a random variable taking only countably many values and having a finite expected value, then
$\mathbf{P}\{X \geq \mathbf{E}[X]\} > 0$, and $\mathbf{P}\{X \leq \mathbf{E}[X]\} > 0$.

The proof is left as an exercise.

✍ **Exercise 1.9.14.** Prove the above corollary.

This result can be used along with Lemma 1.9.1 to create a very interesting combinatorial problem solving tool. It is perhaps best demonstrated through an example.

Example 1.9.15

If G is a finite simple graph on a vertex set V and an edge set E, then show that there exists a bipartite subgraph of G which has at least $|E|/2$ many edges in it.

Solution. Given the graph G, we consider coloring its vertices using two colors say red and blue, where the color for every vertex is chosen independently and uniformly (i.e., each vertex has equal probabilities of getting colored red or blue and the colors for different vertices are chosen independently). Now, for any such coloring we construct a bipartite subgraph of G by deleting all the edges which join two vertices of the same color, and retaining all other edges. Let X be the number of edges in such a subgraph. Then X is a random variable taking values in the finite set $\{0, 1, ..., |E|\}$ and hence has a well-defined finite expectation $\mathbf{E}[X]$. Furthermore, since each vertex is assigned colors independently and with equal probabilities, so each edge is retained with a probability $1/2$, thus linearity of expectation implies that $\mathbf{E}[X] = |E|/2$, whence it follows that there exists at least one coloring for which the subgraph obtained by performing the above construction has at least $|E|/2$ many edges, which completes the solution.

□

A more detailed discussion on the probabilistic method will be presented in a later chapter of the book.

1.10 Extremal combinatorics

Extremal combinatorics is a relatively new branch of combinatorics that deals with configurations which are extreme in a certain sense. A typical question in extremal combinatorics would be to ask the maximum or minimum number of elements that a subset of a set can contain, provided it satisfies some constraints. A nice collection of Olympiad problems in extremal combinatorics can be found at https://dgrozev.wordpress.com/category/combinatorics/extremal-combinatorics/.'
More on extremal combinatorics will be discussed later in the text.

1.11 Chapter exercises

1. Is it possible to paint the natural numbers $\mathbf{N}$ using 4 colours such that there do not exist 4 positive integers a, b, c, d of the same colour which satisfy the equation $3a + 3b = 2c + 2d$?

2. A ***star*** is a particular type of bipartite graph in which one of the two vertex sets in the bipartition contains only one vertex. Is it necessary that a star is always a tree?

3. Let $k \leq n$ be natural numbers. Find the number of $k-$ cycles in the complete graph K_n on n vertices.

4. If k, m, n are natural numbers then find the number of $k-$ cycles in the complete bipartite graph $K_{m,n}$.

5. Let there be n knights placed on a chessboard in such a way that each knight attacks exactly m other knights (among the ones' placed on the chessboard following the usual rules of knight's movement). Show that n must be even.

6. In a finite, simple graph G with vertex set V, we say that a vertex v is an *influencer*[6] if $\deg(v) > \deg(w)$ for each neighbour w of v. Let $\text{Star}(G)$ denote the set of all non-isolated influencers of G. Show that
$$|\text{Star}(G)| < \frac{1}{2}|V|.$$
Give an example where one has $\lim_{|V|\to\infty} \dfrac{|\text{Star}(G)|}{|V|} = \dfrac{1}{2}$.

7. (This exercise assumes familiarity with the basics of matrix multiplication.) Let G be a simple graph on n vertices and let A be the adjacency matrix of G. Let $M = A + A^2 + ... + A^{n-1}$. Show that G is connected if and only if for all $i \in \{1, ..., n\}$ and $j \in \{1, ..., n\}$ with $i \neq j$ the (i, j) th entry of M is positive.
(Hint: what can you tell about the (i, j) th entry of A^k if there is a path of length k between the i th and j th vertices.)

8. (IMO 1991, Problem 4) Suppose G is a connected simple graph having k edges. Prove that it is possible to label the edges as $1, 2, ..., k$ in such a way that for each vertex of degree ≥ 2 the greatest common divisor of the label on the edges incident to the vertex is 1.

9. Let G be a simple finite graph on n vertices in which every vertex has degree ≥ 3. Show that G must contain a cycle C such that the number of vertices in C is not divisible by 3.

10.
    i. Show that there is a way in which a knight can traverse a 8×8 chessboard visiting each square of the chessboard exactly once.
    ii. Using part (i) or otherwise show that the maximum number of pairwise non-attacking knights that can be placed on a 8×8 chessboard is 32. For $n \in \mathbf{N}$ find the maximum number of pairwise non-attacking knights which can be placed on a $n \times n$ chessboard.

[6]the terminology *influencer* was suggested by professor Terence Tao as a gender-neutral and modern substitute for the term *king* which was used in the original instance of the problem (see [6])

iii. (Proposed by Kada Williams) Find the maximum number of knights which can be placed on a 8×8 chessboard in such a fashion that each knight on the chessboard attacks at most 1 other knight.

11. As legend goes once upon a time there lived a king in Babylon who feared being hit and looted by the neighbouring state. To hide some of the most precious gemstones he sent n boxes each containing n gemstones of a particular variant which was amongst the most precious gems across the world, to his most trusted minister's house asking him to take care of them until the king returns to get them back. The minister could not control his greed and opened the boxes removed some of the gemstones from some of these boxes replacing the removed gemstone with a fake one which despite appearing original to the naked eye could easily be detected by any expert. After several days the king returned and brought with him an expert to validate the quality of the gemstones and make sure that the minister has kept his word. The expert was asked to open the boxes one by one and choose one gemstone at random from these boxes and validate its quality. What is the probability that the king is unable to detect the minister's embezzlement if:

    i. the minister took one gemstone from each of the n boxes;

    ii. the minister took one gemstone from each of $\lfloor n/2 \rfloor$ many boxes keeping the remaining $n - \lfloor n/2 \rfloor$ many boxes untouched;

    iii. all the gemstones from just one box keeping other boxes untouched;

    iv. $\lfloor n/2 \rfloor$ many gemstones from one of the boxes, keeping all other boxes untouched.

    (Moral: excessive greed can significantly increase the chances of an upcoming failure.)

12. Suppose we have n symbols $*_1 \ldots\ldots, *_n$, and we want to form a word of length N using these symbols in such a way that no two consecutive $*_1$ s appear in this word. Find the number of such words which can be formed.

13. If $n \in \mathbf{N}$ is a natural number then prove that $n+1$ is a divisor of $\binom{2n}{n}$. [7] A ***Dyck word*** of length $2n$ is a string of n many $\times$ and n many $\circ$ such that no initial segment has more $\times$ than $\circ$. Find the number of Dyck words of length $2n$.

14. For every natural number n, let u_n be the number of walks on $\mathbf{Z}$ starting at 0 at time 0, moving ± 1 step at each time point and ending at 0 at time $2n$ and staying non-negative valued at all time points between time 0 and $2n$. Set $u_0 := 1$. Show that for all positive integers n one has:

$$u_{n+1} = u_0 u_n + u_1 u_{n-1} + \ldots. + u_{n-1} u_1 + u_n u_0.$$

15. If n and k are two positive integers, then find the number of ways of painting the vertices of a n-gon using k many colours, such that no two adjacent vertices receive the same colour?

16. (China West, 2002) Let $A_1, \ldots\ldots, A_{n+1}$ be disjoint subsets of $\{1, \ldots.., n\}$. Prove that there exists non-empty disjoint subsets I, J of $\{1, \ldots.., n+1\}$ such that

$$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j.$$

[7] The number $\dfrac{\binom{2n}{n}}{n+1}$ is called the n ***th Catlan number*** and it plays a central role in mathematics.

17. (Slovak mathematics competition, 2004.) Given 1001 rectangles whose length and width $\in \{1, 2, ..., 1000\}$, prove that there exists at least three of them A, B, C, such that A fits inside B, and B fits inside C.

18. (Romania TST, 2005.) Let $n \in \mathbf{N}$. Let $\mathcal{A}$ be a set of $n^2 + 1$ many positive integers such that for any $(n+1)$ element subset $\mathcal{M} \subseteq \mathcal{A}$, there are elements $a \in \mathcal{M}$, and $b \in \mathcal{M}$ such that $a \neq b$ and $a|b$. Prove that there are $n+1$ many pairwise distinct elements $\ell_1, \ell_2, ..., \ell_{n+1}$ of $\mathcal{A}$, such that $\ell_i | \ell_{i+1}$ $\forall i \in \{1, 2, ..., n\}$.

19. Using a probabilisitic argument show that if n is a positive integer, then the following identity holds
$$\sum_{k=0}^{n} \left(\frac{k \cdot k!}{n^k} \cdot \binom{n}{k} \right) = n.$$

20. If n, k are positive integers such that $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then prove that $R(k,k) > n$, where as usual $R(k,k)$ denotes the (k,k)th Ramsey number.

21. (Taiwan National Mathematical Olympiad, 2006)[8] There are 94 safes and 94 keys, one key for each safe and also assume that each safe has its own unique key. We place randomly one key into each safe, and then we choose 92 safes uniformly at random, and lock them. What is the probability that we can open all the safes with the two keys in the two remaining safes?
Solve the more general problem: suppose there are a total of n safes and n keys (one key for each safe and assume that no key can open more than one safe), suppose the keys are randomly placed in these safes with one key in each safe and then $n-k$ of them are uniformly chosen at random and locked. What is the probability that using the keys available in the k unlocked safes, you can open all the n safes?

* * * * *

[8] This exercise assumes familiarity with the discrete uniform probability distribution (see here).